Notes for *Functional Analysis* (CIMAT, Fall 2020)

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Contents

Notations and conventions

- $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ are respectively the set of nonnegative integers, integers, rational numbers, reals and complex numbers. $\mathbb{F}$ denotes either the field of real numbers or the field of complex numbers. Vector spaces are always on $\mathbb{F}$.

- For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ and $a = (a_1, \dots, a_n) \in \mathbb{R}^n$, we set $|\alpha| = \sum \alpha_j$, $\alpha! = \prod \alpha_j!$, $a^\alpha = \prod a_j^{\alpha_j}$ and $\partial^\alpha = \frac{\partial^{\alpha_1+\cdots+\alpha_n}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$, where $(x_1, \dots, x_n)$ is the standard coordinates on $\mathbb{R}^n$. Also, $0^\alpha := 1$ and $\partial^0 := \mathrm{id}$ if $\alpha = (0, \dots, 0)$.

- Neighborhoods are always open.

- Topological vector spaces (TVS) are assumed to be Hausdorff. LCTVS stands for "locally convex topological vector space".

- "Subspace" always means in the sense of topology, namely a subset with subspace topology. If the subspace is also closed under the vector space structure we use the phrase "linear subspace".

- The domain, kernel (null space) and range of a linear map A are denoted by Dom_A, Ker_A and Ran_A.

- $C(X)$, X compact topological space, is the algebra of continuous functions $f : X \to \mathbb{C}$, equipped with the uniform norm $\|f\| := \sup_{x\in X} |f(x)|$. $C(X; \mathbb{R})$ is the real version.

- $C^k(U)$, $k \in \{0, 1, \dots, \infty\}$, $U \subseteq \mathbb{R}^n$ open, is the set of functions $f : U \to \mathbb{C}$ which has continuous derivative $\partial^\alpha f$ up to total order k, namely $|\alpha| \le k$.

- $C^k(\overline{U})$, $k \in \{0, 1, \dots, \infty\}$, $U \subseteq \mathbb{R}^n$ open, is the set of functions $f : \overline{U} \to \mathbb{C}$ which admit an extension to a C^k function on a neighborhood of $\overline{U}$. Alternatively, by a classical theorem of Borel [Lee, page 27], it is exactly the set of $C^k(U)$ functions f such that each partial derivative $f^{(\alpha)}$, $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, $\sum \alpha_j \le k$, admits a continuous extension to $\overline{U}$.

- Sequence spaces l^p, c, c_0, c_{00} are introduced in Example 1.

- If α is a linear functional on a vector space X and $x \in X$ then $\alpha(x)$ is sometimes denoted by $\langle x, \alpha \rangle$.

- The term "operator", unless otherwise stated, refers to a continuous linear map.

Chapter 1

What is this course about?

Mathematical analysis studies the properties (say differentiability, integrability, Fourier harmonics, etc.) of functions living on topological spaces (say open subsets of $\mathbb{R}^n$). This study can be done at two levels:

1. *Personal life.* To study each function separately. This is called **hard analysis**.

2. *Social life.* To study spaces of functions. This is called **soft analysis**, **functional analysis** or **infinite dimensional analysis**. Almost always these spaces of functions turn out to be infinite-dimensional vector spaces, and since mathematical analysis is about limiting phenomena, these vector spaces are equipped with a structure to measure nearness, namely a norm, a family of norms, a metric or more generally a topology. If the topology is complete (namely Cauchy sequences converge) then geometric intuitions can be developed. Sometimes these spaces are even closed under pointwise multiplication, so form an algebra. (For example, the vector space of continuous complex-valued functions f living on a compact topological space X and equipped with the uniform norm $\|f\| = \sup_{x\in X} |f(x)|$ is an algebra.) This way, functional analysis is a common ground for analysis, topology, geometry and algebra.

In a real problem these two aspects are interwoven. This course emphasizes the soft aspects but will try to enlighten the connections to hard ones.

Functional analysis is a vast subject. Here are some aspects of it:

- Linear algebra is the study of finite dimensional vector spaces. It is a well-developed subject with many highlights such as the theory of determinants, Jordan structure theorem for linear maps, duality theory (for example: $Ax = y$ is solvable if and only if y is orthogonal to the kernel of A^*), Fredholm alternative (A square matrix is either invertible or it is neither injective nor surjective.), etc. On a finite dimensional vector space there is only one Hausdorff topology compatible with the linear structure (Theorem 2.(1) and Chapter 3.(26)). This is why topological issues does not appear in linear algebra. However, topology is an important aspect of infinite dimensional analysis. One mission of functional analysis is to generalize the fundamental theorems of linear algebra to infinite-dimensional setting. For example, spectral theory generalizes principal axis theorem of normal matrices (Chapters 10-11); there is a well-developed duality theory (Chapters 4-7); there are theories for trace and determinant [Lax, chapter 30]; there is a Fredholm alternative (Theorem 61), etc. In the same way, functional analysis generalizes some fundamental theorems of differential and integral calculus on finite dimensional Euclidean spaces $\mathbb{R}^n$ to infinitely many dimensions. For example, the implicit function theorem and the exponential matrix solution $x(t) = e^{At}x(0)$ for the first order linear constant-coefficient system of ordinary differential equations $dx/dt = Ax$ are generalized in Theorems 79 and 82, respectively. Not all theorems of functional analysis are of this sort. For example, the theory of compact and Fredholm operators (Chapter 8) has no finite dimensional counterpart.

- Continuous linear maps between topological vector spaces of functions are called operators; their study, the so-called **operator theory**, is a branch of functional analysis [Alp, Pea]. To get deep results one is usually directed to study specific classes of operators, for example compacts and Fredholms (Chapter 8), integral and pseudodifferential operators ([Grf, chapters 4,6,8][Shu, Tay-PDO]), self-adjoints and normals (Chapters 10-11), isometries and dilations (or contractions) [SFBK], shifts [Nik], Toeplitz operators [Dou, chapter 7][BS, BG, Upm], Hankel operators [Zhu-OT, chapters 7, 9], etc. Generally speaking, the structure theory for a class of operators is called the **spectral theory** for that class of operators; however, this phrase usually refers to the structure theory of normal operators on Hilbert spaces (Chapters 9-11).

- The most important slogan of functional analysis is that continuous linear functionals can be used to detect many notions in functions spaces. For example, they can be used to check boundedness of subsets, density of linear subspaces, surjectivity and range-closedness of operators, etc. Theorem 52 gathers many instances of such phenomena.

- One way to grasp the taste of a branch of mathematics is through the deep theorems proved there. Here is one in operator theory. Beurling characterized all closed linear subspaces of $l^2(\mathbb{N})$ (the space of square-summable sequences of complex numbers) which are invariant under the shift operator $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $(a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots)$. The final answer is extremely hard to express without using a particular function theory specifically developed for this purpose [Rud-RCA, 17.23]. There are very few operators whose lattice of invariant closed linear subspaces is completely known. It is an open problem, the so-called **invariant subspace problem**, whether every operator on a separable infinite-dimensional Hilbert space has a nontrivial invariant closed linear subspace.

Chapters 2-7 constitute the core of this course. Later chapters discuss various topics. We are going to start by introducing several general classes of spaces of functions appearing in mathematical analysis: topological vector spaces, normed vector spaces, Banach spaces, reflexive Banach spaces, uniformly convex Banach spaces and Hilbert spaces, in decreasing order of generality. Only for Hilbert spaces one can develop a structure theory.

Chapter 2

Spaces of functions I: Normed vector spaces

References: [Fol, chapter 5][DS, chapter 2][Dou, chapters 1,3][Bre, chapters 2,5].

The simplest spaces of functions which appear in mathematical analysis are those whose topology is induced by a norm. This chapter is devoted to them.

Let $\mathbb{F}$ be either the field of real numbers or the field of complex numbers, and let X be a vector space over $\mathbb{F}$. X is called a **normed vector space** if there exists a unary operation $\|-\| : X \to [0, \infty)$ on X, called a **norm**, which satisfy:

1. Homogeneity: $\|ax\| = |a|\|x\|$ for every $a \in \mathbb{F}$ and $x \in X$.

2. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

3. Positivity: $\|x\| = 0$ implies $x = 0$.

Note that the norm is always finite. The triangle inequality shows that the norm is a continuous function. If we drop the third positivity axiom from the definition of the norm we have a **seminorm**. There is a standard procedure to make a norm out of a seminorm: pass to the quotient $X/\{x \in X : \|x\| = 0\}$, and set the norm of the equivalence class of each $x \in X$ equal to the seminorm of x. A normed vector space can be made a metric space by the distance function $d(x, y) := \|x-y\|$. The distance function in turn induces a topology on X: A basis for the topology is given by the open balls $\{y \in X : d(x, y) < \epsilon\}$, $x \in X$, $\epsilon > 0$. Recall that all metric space topologies are Hausdorff and first countable. When we talk about the topology of a normed vector space we always mean this topology.

Exercise: Two norms $\|-\|_1$ and $\|-\|_2$ on a vector space X are **equivalent** if there exist $C_1, C_2 > 0$ such that $C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1$ for every $x \in X$. Show that: (1) Equivalent norms induce the same topology. (2) l^1, l^2 and l^∞ norms on $\mathbb{C}^m$, given respectively by $\|z\|_1 = \sum |z_j|$, $\|z\|_2 = \sqrt{\sum |z_j|^2}$ and $\|z\|_\infty = \max |z_j|$ for $z = (z_1, \ldots, z_m)$, are all equivalent.

A linear map $T : X \to Y$ between normed vector spaces is called an **operator** if any of the following equivalent conditions holds:

- T is continuous.

- T is continuous at a point.

- T is **bounded** in the sense that it maps bounded subsets to bounded ones; equivalently, $\|Tx\| \le C\|x\|$ for some $C > 0$ and every $x \in X$. (If this happens then the infimum of such C is called the **operator norm** of T, denoted by $\|T\|$; by homogeneity we have $\|T\| = \sup_{x\neq 0}\|Tx\|/\|x\| = \sup_{\|x\|=1}\|Tx\| = \sup_{\|x\|<1}\|Tx\|$. Here we are putting the trivial case $X = \{0\}$ aside.)

The set of all (bounded) operators from normed vector space X to Y is denoted by $B(X;Y)$. (B, L or $\mathcal{L}$ is also used instead of B in the literature.) One can easily check that $B(X;Y)$ is itself a normed vector space via the operator norm. $B(X;X)$ is abbreviated into $B(X)$. The good news that $B(X)$ is an algebra over $\mathbb{F}$ under composition of operators. $B(X;\mathbb{F})$ is abbreviated into X^*, the so-called **dual space** of X. Elements of X^* are called **continuous linear functionals**. A **linear functional** on X is just a linear map $X \to \mathbb{F}$. The **transpose** of an operator $T : X \to Y$ is given by $T^* : Y^* \to X^*$, $\beta \mapsto \beta \circ T$. T is an **isometry** if $\|Tx\| = \|x\|$ for every $x \in X$. An **isometric isomorphism** between normed vector spaces is an isometry which is also a homeomorphism. A **Banach space** is a normed vector space which is complete in the sense that Cauchy sequences converge. Completeness can be stated equivalently by requiring that absolutely convergent series are convergent, namely $\sum x_j$ converges if $\sum \|x_j\| < \infty$. (Exercise: Prove this equivalence. *Hint.* Suppose that absolutely convergent series are convergent and let x_j be a Cauchy sequence. One can find subsequence j_k such that $\|x_{j_k} - x_{j_{k+1}}\| < 2^{-k}$ for each k. Note that a Cauchy sequence is convergent if it has a convergent subsequence.)

Exercise: Consider a linear map $A : \mathbb{R}^n \to \mathbb{R}^m$ given by matrix coefficients $(Ax)_j = \sum_{k=1}^n a_{jk}x_k$. If $\mathbb{R}^n$ and $\mathbb{R}^m$ are equipped with l^p and l^q norms respectively then the operator norm of A is denoted by $\|A\|_{p,q}$. Show that: $\|A\|_{1,1} = \max_k \sum_j |a_{jk}|$, $\|A\|_{\infty,\infty} = \max_j \sum_k |a_{jk}|$, $\|A\|_{1,\infty} = \max_{j,k}|a_{jk}|$, and that $\|A\|_{2,2}$ equals the square root of the greatest eigenvalue of A^tA, where A^t is the transpose matrix. (*Hint.* For $\|A\|_{2,2}$ use Lagrange multiplier theorem along with the identity $\|Ax\|_2^2 = x^tA^tAx$. The interesting point is that computing the norm $\|A\|_{\infty,1}$ is an NP-hard problem [Roh].)

Exercise: Assume measure space (X,μ) and $1 < p < \infty$. (a) Given $\varphi \in L^\infty$, find the transpose of the multiplication operator $M_\varphi : L^p \to L^p$, $f \mapsto f\varphi$. (2) Given $K : X \times X \to \mathbb{C}$ such that the integral operator $T : L^p \to L^p$, $(Tf)(x) = \int K(x,y)f(y)d\mu(y)$ is bounded, find the transpose of T. (*Hint.* Refer Theorem 53 for the dual of L^p spaces.)

Here are some useful constructions:

- If X is a normed vector space then the closure of each linear subspace of X is again a linear subspace. If X is Banach so is the closure.

- If X and Y are normed vector spaces then the Cartesian product $X\times Y$ is also a normed vector space with the norm $\|(x,y)\| = \max(\|x\|,\|y\|)$. Using other norms gives rise to normed spaces with the same topology (Theorem 2.(1)). If X and Y are Banach so is $X \times Y$.

- If X is a normed vector spaces and $Y \subseteq X$ a closed linear subspace then the **quotient** vector space $X/Y = \{x + Y : x \in X\}$ of cosets of Y can be given the norm $\|x + Y\| =$

$\inf_{y\in Y}\|x+y\|$. (If Y is not closed then this is just a seminorm.) If X is Banach so is X/Y.

- Recall the process of the **completion** of metric spaces [Mun, pages 268-72][Kpy, pages 88-92]: For every metric space X there exists a complete metric space $\widetilde{X}$ which contains X as a dense subset; if $\widetilde{X}'$ is another such space then there is an isometric homeomorphism $\widetilde{X}\to\widetilde{X}'$ which is identity on X. If X is a normed vector space then $\widetilde{X}$ inherits a vector space structure from X. Here are two ways for completing normed vector spaces:

  1. $\widetilde{X}$ is the set of all Cauchy sequences of elements of X, modulo the equivalence relation $(x_j)\sim(y_j)$ if and only if $\|x_j-y_j\|\to 0$. The normed vector space structure on $\widetilde{X}$ is given by $(x_j)+(y_j)=(x_j+y_j)$, $a(x_j)=(ax_j)$, $\|(x_j)\|=\lim\|x_j\|$. X sits inside $\widetilde{X}$ diagonally via $x\mapsto(x,x,\ldots)$. Details can be found in [Wei, 4.11].

  2. In the view of Theorem 27, one can identify $\widetilde{X}$ as the closure of $\{\widehat{x}:x\in X\}$ inside the double dual X^{**}.

Exercise: Suppose a normed vector space X, a closed linear subspace $Y\subseteq X$ and the canonical quotient map $\pi:X\to X/Y$. Show that: (1) π is continuous and open. (2) The topology on X/Y induced by the canonical norm coincides with the quotient topology namely $Z\subseteq X/Y$ is open if and only if $\pi^{-1}(Z)\subseteq X$ is open. (3) If X is Banach so is X/Y. (4) X is Banach if and only if both Y and X/Y are so. (*Hint for (3).* Let z_j be a Cauchy sequence in X/Y. After passing to a subsequence one can assume $\|z_j-z_{j+1}\|<2^{-j}$. Then find $x_j\in X$ such that $\pi(x_j)=z_j$ and $\|x_j-x_{j+1}\|<2^{-j+1}$. Continue! Alternatively, show that every absolutely convergent series in X/Y converges. *Hint for the if part in (4).* Let x_j be a Cauchy sequence in X. x_j+Y is Cauchy in X/Y, so $x_j+Y\to x+Y$ for some $x\in X$. Find $y_j\in Y$ such that $\|x_j-x-y_j\|\leq\|x_j-x+Y\|+2^{-j}$. Show that y_j is Cauchy. Continue!)

Example 1. Here are the most important examples of Banach spaces:

1. The vector space $B(X)$ of all bounded functions $f:X\to\mathbb{C}$ on a set X, equipped with the uniform norm $\|f\|:=\sup_{x\in X}|f(x)|$, is Banach. (Exercise: Prove this.) If X has a topology then the linear subspace of continuous functions $BC(X)$ is closed in $B(X)$, hence Banach. When X is a compact topological space then $BC(X)$ equals the vector space $C(X)$ of continuous functions on X.

2. $L^p(X,\mu)$, $1\leq p\leq\infty$, (X,μ) measure space, the vector space of all (equivalence classes of) L^p integrable functions $X\to\mathbb{C}$, equipped with the norm $\|f\|_p:=\left(\int|f|^p d\mu\right)^{1/p}$, is Banach [Fol, 6.6, 6.8]. If μ is the counting measure then $L^p(X,\mu)$ is denoted by $l^p(X)$. l^p stands for $l^p(\mathbb{N})$.

3. $L^p_a(X)=L^p(X,\mu)\cap\{\text{analytic}\}$, $1\leq p\leq\infty$, $X\subseteq\mathbb{C}^m$ open, μ Lebesgue measure, is a closed subspace of $L^p(X,\mu)$ [Zhu-FT, page 42][Hal-S, pages 187-8], hence Banach. These are the **Bergman spaces**. For more examples of Banach spaces of analytic functions appearing in mathematical analysis refer [Zhu-FT].

4. $C_b^k(U)$, $k \in \{0, 1, 2, \ldots\}$, $U \subseteq \mathbb{R}^n$ open, the set of all C^k functions $f : U \to \mathbb{C}$ such that
$$\|f\| := \sum_{\alpha \in \mathbb{N}^n, |\alpha| \le k} \sup_{x \in U} |\partial^\alpha f(x)| < \infty,$$
is Banach. (Multi-index notation is introduced in page 3.)

5. $W^{p,s}(U)$, $1 \le p \le \infty$, $s \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ open, the vector space of all (equivalence classes of) L^p functions $f : U \to \mathbb{C}$ which all their distributional derivatives of total order $\le s$ are represented by L^p functions, equipped with the norm
$$\|f\| := \left(\sum_{\alpha \in \mathbb{N}^n, |\alpha| \le s} \|\partial^\alpha f\|_{L^p(U)}^p \right)^{1/p},$$
is Banach. These are the **Sobolev spaces**.

6. $\Lambda^\alpha([0, 1])$, $\alpha \in (0, 1]$, the vector space of all functions $f : [0, 1] \to \mathbb{C}$ such that
$$\|f\| := |f(0)| + \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^\alpha} : x, y \in [0, 1], x \ne y \right\} < \infty,$$
is Banach. These are **Hölder spaces**. $\Lambda^1([0, 1])$ is the **Lipschitz space**.

7. $B(X; Y)$, X normed vector space, Y Banach space, is Banach. Most importantly, the dual space of every normed vector space is Banach.

8. Let l^∞ be (Banach) space of all bounded sequences of complex numbers, equipped with the supremum norm. The subset c of all convergent sequences is a closed linear subspace of l^∞, hence Banach. The same is true for the subset c_0 of all sequences converging to 0. However, the linear subspace c_{00} of all sequences with finitely many nonzero terms, is *not* Banach: $x_j = (1, 1/2, \ldots, 1/j, 0, 0, \ldots)$ is a sequence of points of c_{00} which is Cauchy but not convergent (to any point of c_{00}).

9. Let U be an open subset of $\mathbb{R}^n$. The vector space of all continuous functions $f : U \to \mathbb{C}$ compactly supported in U, with the norm $\|f\|$ given by the Riemann integral $\int |f| dx$, is *not* Banach. (Exercise: Why?) Its completion can be identified with $L^1(U)$, the normed vector space of Lebesgue integrable functions [Fol, 2.41]. This gives a functional analysis approach to develop Lebesgue measure and integral: First comes the function space L^1, next the Lebesgue integral, and finally the Lebesgue measure. Details can be found in [Lax, appendix A] or [Ped, chapter 6].

More examples of Banach spaces can be found in [DS, chapter 4][LT]. ■

Here is the fundamental theorem on finite dimensional normed vector spaces:

Theorem 2 (Riesz). *Let X be a normed vector space. Then:*

(1) All norms on a finite-dimensional vector space are equivalent.

(2) Every finite-dimensional linear subspace of X is closed. More generally, the sum of a closed linear subspace and a finite-dimensional linear subspace is closed.

(3; Riesz pseudoorthogonality lemma) If Y is a proper closed linear subspace of X then for every $0 < \epsilon < 1$ there exists $x \in X$ such that $\|x\| = 1$ and $\operatorname{dist}(x, Y) > 1 - \epsilon$.

(4) A linear functional on X is continuous if and only if its kernel is closed.

(5) X is locally compact if and only if it is finite-dimensional.

Proof. (1) Let Y be a vector space with basis $e_1, \dots, e_n$. In this proof always consider Y with the topology induced by the norm $\|\sum a_j e_j\|_\infty = \max |a_j|$, $a_j \in \mathbb{F}$. Assuming another norm $\| - \|$ on Y, since $K := \{y \in Y : \|y\|_\infty = 1\}$ is compact by the Heine-Borel theorem and $\| - \| : Y \to [0, \infty)$ is continuous by the triangle inequality, it follows that $\| - \|$, restricted to K, takes values in a finite interval $[C_1, C_2]$. By homogeneity, $C_1 \|y\|_\infty \leq \|y\| \leq C_2 \|y\|_\infty$ for every $y \in Y$.

(2) Let Y be a finite-dimensional linear subspace of X spanned by linearly independent elements $e_1, \dots, e_n$. Assume that the sequence $\sum_{i=1}^n a_{ji} e_i$, $a_{ji} \in \mathbb{F}$, converges $x \in X$. By the proof of (1), a_{ji} is Cauchy for each i, hence converges some b_i. Therefore $x = \sum b_i e_i \in Y$. The second statement can be easily reduced to the first by moding out the closed linear subspace. (Refer to the first part of Section 3.1.(27), where by the way, another proof for the first statement is given for a larger class of spaces X.) Here is another proof for the second statement using the techniques of Chapter 4. By induction it suffices to show that each $Y + \mathbb{F}x$, $x \in X \setminus Y$, is closed. By Theorem 25.(3) there exists a continuous linear functional α on X such that $\alpha|_Y \equiv 0$ and $\alpha(x) \neq 0$. If $y_j + a_j x$, where $y_j \in T$ and $a_j \in \mathbb{F}$, is a sequence of points in $Y + \mathbb{F}x$ converging $z \in X$ then applying α to this sequence implies that a_j is convergent, so y_j is convergent, hence $x \in Y + \mathbb{F}x$.

(3) Choose $z \in X \setminus Y$. Since Y is closed it follows that $\delta := \operatorname{dist}(z, Y) > 0$. Choose a sequence y_j in Y such that $\operatorname{dist}(z, y_j) \to \delta$. Setting $x_j := (y_j - z)/\|y_j - z\|$ we have $\|x_j\| = 1$ and

$$\operatorname{dist}(x_j, Y) = \inf_{y \in Y} \|x_j - y\| = \inf_{y \in Y} \left\| \frac{y_j - z}{\|y_j - z\|} - y \right\| = \frac{\inf_{y \in Y} \|y_j - z - \|y_j - z\| y\|}{\|y_j - z\|} = \frac{\delta}{\|y_j - z\|},$$

approaches 1 as $j \to \infty$. Therefore, for every $\epsilon > 0$, choosing j large enough, we have $\operatorname{dist}(x_j, Y) \in (1 - \epsilon, 1]$.

(4) Only if part is trivial. To prove the if part, contrapositively, assume a linear functional α on X with closed kernel and a sequence $(x_j)_{j \geq 1}$ of norm-1 elements in X such that $|\alpha(x_j)| > j$. The sequence $y_j := x_1 - \alpha(x_1)/\alpha(x_j) x_j$ is in the kernel of α and $y_j \to x_1$, so $x_1 \in \operatorname{Ker}_\alpha$. This contradicts $|\alpha(x_1)| > 1$. Other proofs: The proof of Theorem 11 (for Hilbert spaces X), Section 3.1.(11) (for topological vector spaces X), the proof of Theorem 25.(5) neglecting the first line (for locally convex topological vector spaces X).

(5) If part is by the famous Heine-Borel theorem that compact subsets of $\mathbb{R}^n$ are exactly those which are both closed and bounded. Assuming that X is infinite-dimensional we need to show that the closed unit ball B of X is not compact. By (2,3), inductively construct sequence x_j in (the boundary of) B such that $\|x_j - x_k\| > 1/2$ whenever $j \neq k$. Such sequence has no convergent subsequence. *Another argument.* If B is compact then there exists finitely many point $x_1, \dots, x_n$ in B such that $B \subseteq \bigcup x_j + \frac{1}{2} B$. Let Y be the linear space of x_j. Then Y is closed by (2), so we have the canonical map $\pi : X \to X/Y$

between normed vector spaces. Then $\pi(B) \subseteq \frac{1}{2}\pi(B)$, so $\pi(B) = 0$, namely $B \subseteq Y$. This can not happen if X is infinite-dimensional. ■

Exercise: Justify the appellation "pseudoorthogonality lemma".

2.1 Open mapping and closed graph theorems, Uniform boundedness principle

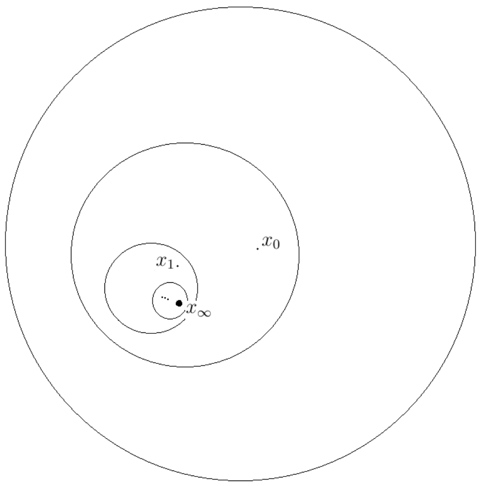
The proof of the fundamental results in the title of this section is based on the following point-set topology result:

Theorem 3 (Baire category theorem)**.** *Let X be complete metric space or a locally compact Hausdorff topological space. Then:*

(1) Countable intersections of dense open subsets is dense.

(2) Countable unions of nowhere dense subsets[1] *has empty interior, so can not be the whole space.*

Proof. (1) First assume X to be a complete metric space. Let $(U_j)_{j\geq 1}$ be a sequence of dense opens of X. Assuming an arbitrary nonempty open ball $B_{r_0}(x_0) \subseteq X$ of radius $r_0 \in (0,1)$ around $x_0 \in X$, we need to find a point in $B_{r_0}(x_0) \cap \bigcap U_j$. Since $B_{r_0}(x_0) \cap U_1$ is nonempty open it contains an open ball $B_{r_1}(x_1)$ of radius $r_1 \in (0, 1/2)$ around $x_1 \in X$. Similarly, $B_{r_1}(x_1) \cap U_2$ contains an open ball $B_{r_2}(x_2)$ of radius $r_2 \in (0, 1/4)$ around $x_2 \in X$. Inductively, one can find sequences $x_j \in X$ and $r_j \in (0, 2^{-j})$ such that $B_{r_{j-1}}(x_{j-1}) \cap U_j$ contains $B_{r_j}(x_j)$. Clearly, x_j is a Cauchy sequence, so converges to some $x_\infty \in X$. By our construction and the triangle inequality we have $x_\infty \in B_{r_0}(x_0) \cap \bigcap U_j$.



The proof for locally compact Hausdorff space X is similar. This time instead of balls $B_{r_j}(x_j)$ we will have nonempty opens B_j resulting from local compactness: $\overline{B_j}$ is compact and contained in $B_{j-1} \cap U_j$. Note that $\bigcap \overline{B_j}$, clearly a subset of $B_0 \cap \bigcap U_j$, is nonempty because of compactness.

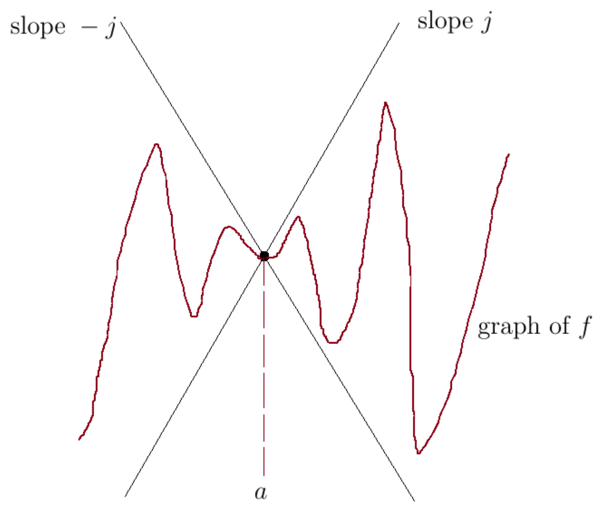
[1] A subset Y of a topological space X is called **nowhere dense** if the interior of the closure of Y is empty; in other words, every nonempty open subset of X has a point not in $\overline{Y}$.

(1)⇒(2) Let Y_j be a sequence of nowhere dense subsets of X. Then $X \setminus \overline{Y_j}$ is a sequence of dense open subsets of X, so $X \setminus \bigcup \overline{Y_j} = \bigcap X \setminus \overline{Y_j}$ is dense. Therefore $\bigcup Y_j$ has no interior. ■

Application 4. *There exists a continuous function* $[0,1] \to \mathbb{R}$ *which is not differentiable at any point; in fact, nowhere differentiable functions are dense in* $C([0,1];\mathbb{R})$ *with respect to the uniform topology.*

Proof. Let $X := C([0,1];\mathbb{R})$ be the vector space of continuous functions $[0,1] \to \mathbb{R}$, equipped the with uniform topology. If $f \in X$ is differentiable at $x = a$ then $|f(x) - f(a)| \leq (|f'(a)| + 1)|x - a|$ on some neighborhood of a; since the graph of f is compact outside this neighborhood it follows that $|f(x) - f(a)| \leq K|x - a|$ on whole $[0,1]$ for some large enough K. Contrapositively, this shows that the set of nowhere differentiable functions in X contains $\bigcap_{j=1}^{\infty} X \setminus Y_j$ where

$$Y_j = \{f \in X : |f(x) - f(a)| \leq j|x - a|, \exists a \in [0,1] \forall x \in [0,1]\}.$$



The intuitive picture about elements $f \in Y_j$ is that their graph are confined in the biconical region with vertex centered at some $(a, f(a))$ and border slopes $\pm j$. By Baire category theorem we need to show that each $X \setminus Y_j$ is open and dense. To show that Y_j is closed, assume a sequence f_n in Y_j which converges $f \in X$. Since $f_n \in Y_j$ there exists a_n such that $|f_n(x) - f_n(a_n)| \leq j|x - a_n|$ for every x. By Bolzano-Weiertrass theorem, after passing to a subsequence one can assume that a_n converges some a. Since the convergence of f_n is uniform it follows that $f_n(a_n) \to f(a)$ and we have

$$|f(x) - f(a)| = \lim_n |f_n(x) - f_n(a_n)| \leq \lim_n j|x - a_n| = j|x - a|, \quad \forall x \in [0,1],$$

hence $f \in Y_j$. To prove the denseness of $X \setminus Y_j$, fixing $f \in X$ and $\epsilon > 0$ we should find some $h \in X$ with $\|h - f\| < \epsilon$ and $h \notin Y_j$. By the uniform continuity of f one can find a piecewise linear function (with finitely many slopes) in any neighborhood of f, so without loss of generality we can assume f is piecewise linear. The idea is to mount a sharp sawtooth function on f. Let $M \geq 0$ be the maximum slope of f, measured in absolute value. (That is if f has slopes $s_1, \ldots, s_n$ then $M := \max\{|s_1|, \ldots, |s_n|\}$.) Choose an integer N strictly larger that $(M + j)/\epsilon$, and construct the sawtooth function $g \in X$ with vertices $g(j/N) = (-1)^j$ for $j = 0, \ldots, N$. We assert that $h := f + \epsilon g/2 \in X$ works as our desired function. Clearly, $\|h - f\| = \epsilon/2 < \epsilon$. On the other hand, h is piecewise linear, and the absolute values of the slopes of h exceed

$$\left| M - \frac{\epsilon}{2} \frac{2}{1/N} \right| = |M - \epsilon N| > j.$$

Therefore $h \notin Y_j$. ■

Here are some concrete examples of continuous nowhere differentiable functions:

- $\sum_{j \geq 0} 2^{-j} \exp(\sqrt{-1}3^j x)$ [Grf, 3.7.3][Har].
- $\sum_{j \geq 0} 10^{-j} \{10^j x\}$, where $\{x\}$ denotes the distance of the real number x to the nearest integer [DiB, page 228].

Exercise: The vector space dimension of a Banach space can not be $\aleph_0$, the cardinality of integers. (*Hint.* Use the Baire category theorem along with the fact that finite dimensional linear subspaces of a normed vector space are closed.)

Theorem 5 (Banach)**.** *(1- Uniform boundedness principle) A family $T_\alpha : X \to Y$, $\alpha \in A$, of bounded operators from Banach space X to normed vector space Y is uniformly equibounded (namely $\sup_{\alpha \in A} \|T_\alpha\| < \infty$) if and only if it is pointwisely equibounded (namely $\sup_{\alpha \in A} \|T_\alpha x\| < \infty$ for every $x \in X$).*

(2- Open mapping theorem) A bounded operator between Banach spaces is surjective if and only if it is open (namely maps opens to opens).

(2′- Inverse mapping theorem) A bounded operator between Banach spaces is bijective if and only if it a homeomorphism.

(3- Closed graph theorem) A linear map $T : X \to Y$ between Banach spaces is continuous (namely for every sequence x_j in X such that $x_j \to x$ it is the case that $Tx_j \to Tx$) if and only if its graph $\mathcal{G}_T := \{(x, Tx) : x \in X\}$ is closed in $X \times Y$ (namely for every sequence x_j in X such that $x_j \to x$ and $Tx_j \to y$ it is the case that $Tx = y$).

A family of bounded operators from a Banach space to a normed vector space is uniformly equibounded if and only if it is pointwisely equibounded.

A bounded operator between Banach spaces is surjective (respectively, bijective) if and only if it open (respectively, a homeomorphism).

To show that a linear map T between Banach space is continuous it is enough to check that $x_j \to 0$ and $Tx_j \to y$ implies $y = 0$.

Proof. (1) Only if part is trivial. For the other direction, since X is the union of closed subsets $X_j := \{x \in X : \sup \|T_\alpha x\| \leq j\}$, $j = 1, 2, \ldots$, by Baire category theorem some X_j contains a nonempty open ball $B_\epsilon(x_0) = \{x \in X : \|x - x_0\| < \epsilon\}$. This means that $\|x - x_0\| < \epsilon$ implies $\sup \|T_\alpha x\| \leq j$, hence $\sup \|T_\alpha(x - x_0)\| \leq 2j$. Therefore, $\sup \|T_\alpha\| \leq 2j/\epsilon$.

(2) Let $T : X \to Y$ be a bounded operator between Banach spaces. For any positive real r let X_r be the open ball of radius r in X around the origin; similarly define Y_r. Since T is linear and all nonempty open balls in a normed vector space are homeomorphic it follows that T being open is equivalent to TX_1 containing some Y_r. If part is now trivial. Conversely, assume that T is surjective. Therefore $Y = \bigcup_{j \in \mathbb{N}} TX_j$. Since all TX_j are homeomorphic it follows by Baire category theorem that $\overline{TX_1} \supseteq B_r(y_0)$, open ball in Y of radius $r > 0$ around some y_0. From this, just using the linearity of T, we will show that $\overline{TX_1} \supseteq Y_{r/4}$, as follows. Since $y_0 \in \overline{TX_1}$ it follows that there exists $x_1 \in X_1$ such that $\|y_0 - y_1\| < r/2$ where $y_1 = Tx_1$. Since $X_1 \subseteq B_2(x_1)$ and $B_r(y_0) \supseteq B_{r/2}(y_1)$ it follows that $\overline{TB_2(x_1)} \supseteq B_{r/2}(y_1)$. By linearity $\overline{TX_2} \supseteq Y_{r/2}$, or equivalently, $\overline{TX_1} \supseteq Y_{r/4}$. Replacing r with $4r$ we deduce that $\overline{TX_1} \supseteq Y_r$, or more generally $\overline{TX_{2^{-j}}}$ contains $Y_{2^{-j}r}$ for every $j \in \mathbb{N}$. We assert that $TX_1 \supseteq Y_{r/2}$. Assume $y \in Y_{r/2}$. Find $x_1 \in X_{1/2}$ such that $\|y - Tx_1\| < r/4$. Since $y - Tx_1 \in Y_{r/4}$ one can find $x_2 \in X_{1/4}$ such that $\|y - Tx_1 - Tx_2\| < r/8$. Continuing this process we find a sequence $x_j \in X_{2^{-j}}$ such that $\|y - T\sum_{k=1}^{j} x_k\| < r2^{-j-1}$. The series $\sum x_j$ is absolutely convergent, so it converges some $x \in X$ with $\|x\| \leq \sum \|x_j\| \leq \sum 2^{-j} < 1$. We found $x \in X_1$ with $y = Tx$.

(2)⇒(2′) If part is trivial. Only if part is immediate from (2).

(2′)⇒(2) Let $T : X \to Y$ be a surjective bounded operator between Banach spaces. Since T is continuous it follows that $\mathrm{Ker}_T \subseteq X$ is closed. T equals the canonical quotient map $X \to X/\mathrm{Ker}_T$ composed with the bijective linear map $S : X/\mathrm{Ker}_T \to Y$, $x + \mathrm{Ker}_T \mapsto Tx$. For every $x \in X$, the computation

$$\|S(x + \mathrm{Ker}_T)\| = \inf_{x' \in x + \mathrm{Ker}_T} \|Tx'\| \leq \|T\| \inf_{x' \in x + \mathrm{Ker}_T} \|x'\| \leq \|T\|\|x + \mathrm{Ker}_T\|$$

shows that S is bounded. Therefore, S is a homeomorphism by (2). T is open because one can easily show that every canonical quotient map between normed vector spaces is open.

(2′)⇒(3) Consider the bijective linear map $S : X \to \mathcal{G}_T$, $x \mapsto (x, Tx)$. Note that $\mathcal{G}_T$ is a closed linear subspace of a Banach space, so is itself Banach. Applying (2′) to the (continuous) set-theoretic inverse $\pi_1 : \mathcal{G}_T \to X$ of S implies that S is continuous. The continuity of T follows. ■

To see this fundamental theorem of Banach in more generality refer Remark 24. If the Banachness assumption is dropped from any of the statements in Theorem 5 then the conclusion does not hold. (For examples refer [Fol, exercises 29–31] or one of the exericses in page 15.

Here is an application of the uniform boundedness principle to Fourier analysis.

Application 6 (duBois Reymond)**.** *There exist a continuous periodic function $\mathbb{R} \to \mathbb{C}$ whose Fourier series diverges at a point.*

Proof. We will prove the existence of some $f \in C([0,1])$ whose Fourier series diverges at the origin. Recall that the partial sums of the Fourier series of any $f \in C([0,1])$ is given by

$$S_j(x) = \sum_{|k|\leq j} c_k e^{2\pi\sqrt{-1}kx} \quad \text{where} \quad c_k = \int_0^1 f(x)e^{-2\pi\sqrt{-1}kx}dx, \quad k \in \mathbb{Z}.$$

Therefore at $x = 0$ we have

$$S_j := S_j(0) = \int_0^1 f(x)D_j(x)dx,$$

where the Dirichlet kernel $D_j(x)$ is given by

$$D_j(x) = \sum_{|k|\leq j} e^{-2\pi\sqrt{-1}kx} = \begin{cases} \frac{\sin(\pi(2j+1)x)}{\sin(\pi x)}, & 0 < x < 1, \\ 2j+1, & x = 0 \text{ or } x = 1 \end{cases}.$$

Now consider S_j as a sequence of linear maps from Banach space $C([0,1])$ to normed vector space $\mathbb{C}$. Each S_j is a bounded operator whose norm does not exceed $C_j := \int_0^1 |D_j(x)|dx < \infty$. On the other hand, since each $D_j(x)$ has a finite number of simple zeros on $[0,1]$ it follows that its sign function sgn $D_j(x)$ has a finite number of jump discontinuities; therefore for every $\epsilon > 0$, by modifying this sign function on a small neighborhood of each of its discontinuities, one can construct $f \in C([0,1])$ such that $\|f\| = 1$ and $|S_j f| \geq C_j - \epsilon$. We have shown that $\|S_j\| = C_j$. If we prove that $C_j \to \infty$ then by the uniform boundedness principle there exists $f \in C([0,1])$ such that the sequence $|S_j f|$ is unbounded, and we are done. There exist positive constants K_1 and K_2 such that:

$$\begin{aligned} C_j &= 2\int_0^{\frac{1}{2}} \frac{|\sin(\pi(2j+1)x)|}{\sin(\pi x)}dx = 2\int_0^{\frac{1}{2}} \frac{|\sin(\pi(2j+1)x)|}{\pi x}\frac{\pi x}{\sin(\pi x)}dx = \\ &\geq K_1 \int_0^{\frac{1}{2}} \frac{|\sin(\pi(2j+1)x)|}{x}dx = K_1\int_0^{j+\frac{1}{2}} \frac{|\sin(\pi x)|}{x}dx \geq K_1 \sum_{k=0}^{j-1}\int_k^{k+1} \frac{|\sin(\pi x)|}{x}dx = \\ &K_1 \sum_{k=0}^{j-1}\int_0^1 \frac{|\sin(\pi x)|}{x+k}dx \geq K_2 \sum_{k=1}^{j-1}\int_{\frac{1}{4}}^{\frac{3}{4}} \frac{dx}{x+k} = K_2 \sum_{k=1}^{j-1} \log\frac{k+\frac{3}{4}}{k+\frac{1}{4}}. \end{aligned}$$

The last series is asymptotically equivalent to the harmonic series, hence diverges. ∎

Remark 7. For an explicit functions satisfying Application 6 refer [Grf, volume I, exercise 3.3.6]. Kolmogorov constructed an L^1 function whose Fourier series diverges everywhere [Kol][Zyg, volume I, pages 310-4]. Carleson and Hunt proved that the Fourier series of an L^p function, $1 < p < \infty$, converges almost everywhere [Grf, chapter 11]. It is relatively easy to show that the Fourier series of a C^1 function (or even a function of bounded variation) converges everywhere [Apo-A, 11.12][Grf, 3.3.9]. ∎

Exercise: Let $\mathbb{N}_+$ be the set of positive integers, $Y := l^1(\mathbb{N}_+)$ and $X := \{x \in Y : \sum j|x_j| < \infty\}$. Show that: (1) X is not complete. (2) $T : X \to Y$, $(Tx)_j = jx_j$, is bijective, closed, but not bounded. Why the closed graph theorem does not apply? (3) The set-theoretic inverse $S : Y \to X$ of T is bounded, neither open nor a homeomorphism. Why the open mapping theorem or the inverse mapping theorem does not apply? (*Hint for (1).* By truncating series show that $X \subsetneq Y$ is dense. *Hint for (2).* To show that T is not bounded consider $x = (q^j)$, $q \in \mathbb{R}$, or x given by $x_n = 1$ and zero everywhere.)

Exercise: Prove the following if you are familiar with Hilbert spaces. A linear map $T : X \to X$ on Hilbert space X which satisfies $\langle Tx, y\rangle = \langle x, Ty\rangle$ for every $x, y \in X$ (so-called self-adjoint operator) is continuous. (*Hint.* Use the closed graph theorem.)

Exercise: Let $\|-\|_1$ and $\|-\|_2$ be two norms on a vector space which makes it Banach. If there exists $C > 0$ such that $\|x\|_1 \le C\|x\|_2$ for every $x \in X$ then there exists $D > 0$ such that $\|x\|_2 \le D\|x\|_1$ for every $x \in X$, so the norms are equivalent. (*Hint.* Use the inverse mapping theorem.)

Exercise: Let $T_j : X \to Y$ be a sequence of bounded operators between Banach spaces such that $T_j x$ converges for every $x \in X$. Then the limit map $T : X \to Y$, $x \mapsto \lim T_j x$, is a bounded operator. This is called **Banach-Steinhaus Theorem**. (*Hint.* Use the uniform boundedness principle.)

Exercise: A bilinear map $B : X \times Y \to Z$, with X Banach and Y, Z normed vector space, is continuous if and only if it is separately continuous (namely $B(x, -)$ and $B(-, y)$ are continuous for every $x \in X$ and $y \in Y$) if and only if there exists $C > 0$ such that $\|B(x, y)\| \le C\|x\|\|y\|$ for every $x \in X$ and $y \in Y$. (*Hint.* To deduce the last statement from the second apply the uniform boundedness principle to the family of bounded operators $T_y := B(-, y)/\|y\|$, $y \in Y \setminus \{0\}$.)

Exercise: Let $L^p_a(\mathbb{D})$ be the Bergman space on the unit disk of the complex plane (Example 1). Justify the use of closed graph theorem in the following excerpt from [DS, page 61]: "The problem here is to characterize those finite Borel measures μ on $\mathbb{D}$ with the property that $\int |f|^p d\mu < \infty$ for every $f \in L^2_a(\mathbb{D})$. If μ is any such measure it follows from the closed graph theorem that $\int |f|^p d\mu \le K\|f\|^p_p$ for some constant $K > 0$ depending only on p."

The following strong version of injectivity of operators is very important in operator theory:

Theorem 8. *An operator $T : X \to Y$ between Banach spaces is injective and range-closed if and only if it is* **bounded from below** *in the sense that $\|Tx\| \ge C\|x\|$ for some $C > 0$ and every $x \in X$.*

Proof. Let T be bounded from below. T is clearly injective. To prove that Ran_T is closed assume a sequence x_j in X such that Tx_j converges some $y \in Y$. Then Tx_j is Cauchy, so by the bounded-from-below inequality x_j is also Cauchy. Therefore, x_j converges some $x \in X$, and by continuity $Tx = y$. This proves that Ran_T is closed. Conversely, let T be injective and range-closed. Then the inverse mapping theorem applied to $X \to \mathrm{Ran}_T$, $x \mapsto Tx$, gives the bounded-from-below inequality. ■

2.2 Hilbert spaces

As usual let $\mathbb{F}$ be either the field of reals or the field of complex numbers, and let X be a vector space over $\mathbb{F}$. X is called a **pre-Hilbert space** if it is equipped with an **inner**

product namely a binary operation $\langle -, - \rangle : X \times X \to \mathbb{F}$ satisfying:

1. $\mathbb{F}$-linearity with respect to the second argument.[2]

2. Conjugate (or Hermitian) symmetry: $\langle x, y \rangle = \overline{\langle y, x \rangle}$ for every $x, y \in X$. (When $\mathbb{F} = \mathbb{R}$ this reduces to the usual symmetry $\langle x, y \rangle = \langle y, x \rangle$.)

3. Positivity: $\langle x, x \rangle \geq 0$ for every $x \in X$, with the equality happening only for $x = 0$.

We develop the theory for complex vector spaces but the theory for real vector spaces is similar. Here are some facts about pre-Hilbert spaces:

- Cauchy-Schwarz inequality:
$$| \langle x, y \rangle | \leq \|x\| \|y\|,$$
with the equality happening exactly when x and y are linearly dependent. Here $\|x\|$ is defined by $\sqrt{\langle x, x \rangle}$, and similarly for $\|y\|$. We will show in the next item that this in fact defines a norm on X.

  (*Proof.* Put the trivial case $y = 0$ aside. Expand $\left\| x - \langle y, x \rangle y / \langle y, y \rangle \right\|^2 \geq 0$. *Another argument.* For real variable t the quadratic expression $\left\| x - ty \right\|^2 = \|y\|^2 t^2 - 2t \mathrm{Re} \langle x, y \rangle + \|x\|^2$ is everywhere nonnegative, so its discriminant $\Delta = 4 \left(\mathrm{Re} \langle x, y \rangle \right)^2 - 4\|x\|^2 \|y\|^2$ is nonnegative, namely $|\mathrm{Re} \langle x, y \rangle | \leq \|x\| \|y\|$. Replacing y by $y \exp(\sqrt{-1}\theta)$, $\theta \in \mathbb{R}$, implies $|\mathrm{Re} \langle x, y \rangle \cos\theta - \mathrm{Im} \langle x, y \rangle \sin\theta| \leq \|x\| \|y\|$. We are done since this is true for every θ.)

- $\|x\| := \sqrt{\langle x, x \rangle}$ is a norm. When we talk about the topology of a pre-Hilbert space we always mean the topology induced by this norm, namely balls $\{y \in X : \|x - y\| < \epsilon\}$, $x \in X$, $\epsilon > 0$, are basic opens.

  (*Proof.* Triangle inequality is immediate from the Cauchy-Schwarz inequality.)

- The inner product is continuous with respect to each of its arguments separately.

  (*Proof.* Immediate from the Cauchy-Schwarz inequality.)

- Parallelogram law:
$$\|x + y\|^2 + \|x - y\|^2 = 2 \left(\|x\|^2 + \|y\|^2 \right).$$

  (*Proof.* Add up two identities $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\mathrm{Re} \langle x, y \rangle$.)

- Polarization identity: The inner product can be reconstructed from the norm it induces, more precisely:
$$\langle x, y \rangle = \begin{cases} \frac{1}{4} \sum_{j=0}^{3} \sqrt{-1}^{j} \left\| \sqrt{-1}^{j} x + y \right\|^2, & \text{complex pre-Hilbert spaces} \\ \frac{1}{4} \left(\|x + y\|^2 - \|x - y\|^2 \right), & \text{real pre-Hilbert spaces} \end{cases}.$$

[2] This is physicists' convention. Mathematicians usually assume linearity with respect to the first argument.

- Jordan-von Neumann theorem: A norm on a vector space is induced by an inner product (namely via $\|x\| = \sqrt{\langle x, x\rangle}$) if and only if it satisfies the parallelogram law.

  (*Proof.* Let the norm $\| - \|$ satisfy the parallelogram identity, and define the inner product by the polarization identity. The only nontrivial thing to check it the linearity of the inner product with respect to its second argument. It is straightforward to show that $\langle x, y\rangle + \langle x, z\rangle = 2\langle x, (y+z)/2\rangle$. Setting $z = 0$ and by induction one can deduce $\langle x, y\rangle + \langle x, z\rangle = \langle x, y+z\rangle$ and $\langle x, ay\rangle = a\langle x, y\rangle$ for $a = 2^{-j}k$ where $j, k \in \mathbb{N}$. Since scalars of the form $2^{-j}k$ are dense among positive reals it follows that $\langle x, ay\rangle = a\langle x, y\rangle$ for every positive real a. That the same identity is true for $a = -1$ and $a = \sqrt{-1}$ is straightforward. Details can be found in [Wei, 1.6].)

Exercise: Prove that there is no inner product on $C([0, 1])$ which induces the uniform norm.

A pre-Hilbert space which is complete with respect to the norm $\|x\| = \sqrt{\langle x, x\rangle}$ induced by its inner product is called a **Hilbert space**. (Completeness means that Cauchy sequences converge.) An **Hilbert spaces isomorphism** is a bijective linear map between Hilbert spaces which preserves the inner product. (Such a map and its inverse are clearly isometries.) Two elements of a Hilbert space are said to be **orthogonal** to each other if their inner product equals zero. For a subset S of a Hilbert space, $S^\perp$, called the **orthogonal complement of** S, is the set of all elements in the Hilbert space which are orthogonal to every element of S. Note that $S^\perp$ is always a closed linear subspace of the ambient Hilbert space.

Here are some easy facts and constructions:

- The closure of a linear subspace of a Hilbert space is again a Hilbert space.

- The Cartesian product of two Hilbert spaces X, Y is a Hilbert space via the inner product $\langle (x_1, y_1), (x_2, y_2)\rangle = \langle x_1, x_2\rangle + \langle y_1, y_2\rangle$. It is usually denoted by $X \oplus Y$.

- Recall the process of completion of normed vector spaces by Cauchy sequences (page 8). If $\widehat{X}$ is the completion of pre-Hilbert space X then $\langle (x_j), (y_j)\rangle = \lim \langle x_j, y_j\rangle$ makes $\widehat{X}$ a Hilbert space. More conceptually, since the inner product on X is a continuous function it can be uniquely extended to an inner product on $\widehat{X}$.

- Suppose Hilbert spaces X, Y over field $\mathbb{F}$, and let Z be their algebraic tensor product, that is the free vector space generated by the set of symbols $\{x \otimes y : x \in X, y \in Y\}$ mod out by the relations

  $$(x + x') \otimes y = x \otimes y + x' \otimes y, \quad x \otimes (y + y') = x \otimes y + x \otimes y', \quad (ax) \otimes y = x \otimes (ay),$$

  where $x, x' \in X$, $y, y' \in Y$, $a \in \mathbb{F}$. Z can be made a pre-Hilbert space by the inner product

  $$\left\langle \sum_j x_j \otimes y_j, \sum_k x_k \otimes y_k \right\rangle = \sum_{j,k} \langle x_j, x_k\rangle \langle y_j, y_k\rangle .$$

  The completion of Z is called the **tensor product** of X, Y, and denoted by $X \otimes Y$. This construction is of fundamental importance in quantum mechanics, basically in the description of the state space of a composite system in terms of the state spaces of its parts [Hall, pages 430, 432, 434]. Here is the construction. Refer [Wei, section 3.4] for more details. Exercise: After learning the notion of orthonormal bases of Hilbert spaces, prove that if $\{x_\alpha : \alpha \in A\}$ and $\{y_\beta : \beta \in B\}$ are orthonormal beses for Hilbert spaces X and Y, respectively, then $\{x_\alpha \otimes \beta : \alpha \in A, \beta \in B\}$ is an orthonormal basis for $X \otimes Y$.

Example 9. Here are the most important examples of Hilbert spaces:

- $\mathbb{C}^n$ is a Hilbert space with inner product $\langle (x_1, \ldots, x_n), (y_1, \ldots, y_n) \rangle = \sum_{j=1}^n \overline{x_j} y_j$.
- $L^2(X, \mu)$, (X, μ) measure space, is a Hilbert space with inner product $\langle f, g \rangle = \int \overline{f} g d\mu$. When μ is the counting measure $L^2(X, \mu) = l^2(X)$. Theorem 21 shows that these latter constitute all Hilbert spaces up to isomorphisms of Hilbert spaces.
- $L^2_a(X) = L^2(X) \cap \{\text{holomorphic}\}$, $X \subseteq \mathbb{C}^m$ open, is a closed subspace of $L^2(X)$, hence a Hilbert space.
- $W^{2,s}(U)$, $s \in \mathbb{N}$, $U \subseteq \mathbb{R}^n$ open, is a Hilbert space with the inner product

$$\langle f, g \rangle = \sum_{\alpha \in \mathbb{N}^n, |\alpha| \leq s} \int \overline{\partial^\alpha f(x)} \partial^\alpha g(x) dx,$$

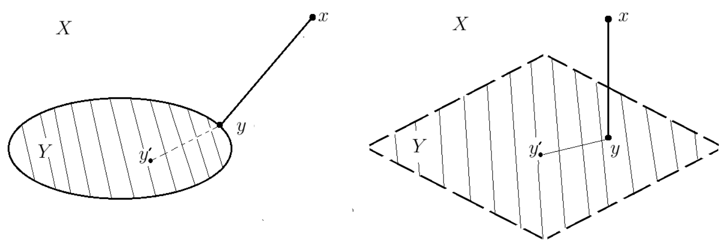
  where dx denotes the Lebesgue measure. (Refer Example 1 for the definition of Sobolev spaces.) ■

Here is the most fundamental variational or geometric property of Hilbert spaces, all the other future theorems are based on:

Theorem 10 (Orthogonal projection)**.** *Let X be a Hilbert space and $Y \subseteq X$ a nonempty closed convex subset. Then:*

(1) For every $x \in X$ there exists a unique $y \in Y$ of shortest distance to x. y is the only element in Y such that $\operatorname{Re} \langle x - y, y' - y \rangle \leq 0$ for every $y' \in Y$.

(2) If Y is furthermore assumed to be a linear subspace then y, obtaind in (1), is the only element in Y such that $x - y \in Y^\perp$. The mapping $P : X \to X$, $x \mapsto y$, is called the **orthogonal projection in X onto Y**. *P is a bounded operator of norm 1, unless $Y = \{0\}$.*



Proof. (1) Let $\delta \geq 0$ be the distance of x to Y, namely $\inf_{y \in Y} \|x - y\|$, and let y_j be a sequence in Y realizing δ, namely $\|x - y_j\| \to \delta$. y_j is a Cauchy sequence because the expression

$$\|y_j - y_k\|^2 = \|(x - y_j) - (x - y_k)\|^2 = 2\|x - y_j\|^2 + 2\|x - y_k\|^2 - 4\|x - (y_j + y_k)/2\|^2 \leq \\ 2\|x - y_j\|^2 + 2\|x - y_k\|^2 - 4\delta^2,$$

can be arbitrary small if j, k are sufficiently large. (We have used the parallelogram identity. $(y_j + y_k)/2 \in Y$ because Y is convex.) Let $y_j \to y$. Then $y \in Y$ because Y is closed. The continuity of the norm implies $\|x - y\| = \delta$. If $\widetilde{y}$ is another point of distance δ to x then again by the parallelogram identity and the convexity of Y we have

$$\|y - \widetilde{y}\|^2 = \|(x - y) - (x - \widetilde{y})\|^2 = 2\|x - y\|^2 + 2\|x - \widetilde{y}\|^2 - 4\left\|x - \frac{y + \widetilde{y}}{2}\right\|^2 \leq 4\delta^2 - 4\delta^2 = 0,$$

hence $y = \widetilde{y}$. We next show that

$$\operatorname{Re}\langle x - y, y' - y\rangle \leq 0, \quad \forall y' \in Y, \tag{2.1}$$

is equivalent to y being a point in Y of shortest distance to x. If (2.1) holds then

$$\|x - y'\|^2 = \|(x - y) - (y' - y)\|^2 = \|x - y\|^2 - 2\operatorname{Re}\langle x - y, y' - y\rangle + \|y' - y\|^2 \geq \|x - y\|^2.$$

For the other direction fix $y' \in Y$. For every real number $t \in [0, 1]$ we have

$$\|x - y\|^2 \leq \|x - (ty' + (1 - t)y)\|^2 = \|(x - y) - t(y' - y)\|^2 = \\ \|x - y\|^2 - 2t\operatorname{Re}\langle x - y, y' - y\rangle + t^2\|y' - y\|^2,$$

or equivalently

$$0 \leq -2\operatorname{Re}\langle x - y, y' - y\rangle + t\|y' - y\|^2, \quad \forall t \in (0, 1]. \tag{2.2}$$

Tending $t \to 0+$ gives (2.1).

(2) We first show that $x - y \in Y^\perp$. Fixing $y' \in Y$, for every real number $t \in \mathbb{R}$ we have

$$\|x - y\|^2 \leq \|x - (ty' + (1 - t)y)\|^2 = \|(x - y) - t(y' - y)\|^2 = \\ \|x - y\|^2 - 2t\operatorname{Re}\langle x - y, y' - y\rangle + t^2\|y' - y\|^2,$$

which happens exactly when $\operatorname{Re}\langle x - y, y' - y\rangle = 0$. Since $y' \in Y$ was arbitrary we have $\operatorname{Re}\langle x - y, y'\rangle = 0$. Replacing y' by $\sqrt{-1}y'$ we have $\operatorname{Im}\langle x - y, y'\rangle = 0$, hence $\langle x - y, y'\rangle = 0$. For uniqueness, set $z := x - y$ and let $\widetilde{y}$ be another element of Y such that $\widetilde{z} := x - \widetilde{y} \in Y^\perp$. Then $y - \widetilde{y} = \widetilde{z} - z$ belongs to $Y \cap Y^\perp = \{0\}$. ■

Compare this proof with the proof of the finite dimensional case given in [Apo-C, volume I, Theorems 15.13–16].

Exercise: Let D be a bounded open subset of the complex plane. Note that then the Bergman space $L^2_a(D)$ contains all polynomial functions. Given arbitrary distinct points $a_j \in D$, $j = 1, \ldots, n$ and arbitrary complex numbers b_j, prove that there exists $f \in L^2_a(D)$ of smallest norm such that $f(a_j) = b_j$ for every j. (See also [DS, Section 1.4].)

Theorem 11 (Riesz representation theorem)**.** *Let X be a Hilbert space. The conjugate-linear mapping $X \to X^*$, $x \mapsto \langle x, - \rangle$, is an isometric isomorphism of normed vector spaces. In other words, for every continuous linear functional α on X there exists a unique vector $x \in X$ such that $\alpha(y) = \langle x, y \rangle$ for every $y \in X$, and $\|\alpha\| = \|x\|$. There is a unique inner product on X^* which induces the operator norm, and it is given by $\langle \langle x, - \rangle, \langle y, - \rangle \rangle_{X^*} = \langle y, x \rangle_X$. Furthermore, X is reflexive (defined in Theorem 27.)*

Proof. For every $x \in X$ let α_x denote $\langle x, - \rangle$. Assume $\alpha \in X^*$, and let Y be the kernel of α. Note that Y is a closed subspace. If α is the zero functional then $x = 0$ works, otherwise since $X = Y + Y^\perp$, one can find a nonzero element $z \in Y^\perp$. Since the linear functional $\beta := \alpha_z$ vanishes wherever α does it follows that $\beta = C\alpha$ for some scalar C. (*Proof.* Choose $z_0 \in X$ such that $\alpha(z_0) = 1$. For each $z \in X$, since α vanishes at $z - \alpha(z)z_0$ it follows that $0 = \beta(z - \alpha(z)z_0) = \beta(z) - \beta(z_0)\alpha(z)$, so $C := \underline{\beta(z_0)}$ works. Q.E.D.) Evaluating $\beta = C\alpha$ at z gives $C = \|z\|^2/\alpha(z)$. This means that $x := \overline{\alpha(z)}\|z\|^{-2}z$ works. Uniqueness: If $\alpha_x = \alpha_{x'}$ then $x - x' \in X^\perp = \{0\}$. Next we compute the operator norm $\|\alpha_x\| = \sup_{\|y\| \le 1} |\langle x, y \rangle|$. By the Cauchy-Schwarz inequality we have $\|\alpha_x\| \le \|x\|$. Setting $y := x/\|x\|$ (in case $x \neq 0$) gives $\|\alpha_x\| \le \|x\|$, hence $\|\alpha_x\| = \|x\|$. This latter equation persists when $x = 0$. Clearly, $\langle \alpha_x, \alpha_y \rangle := \langle y, x \rangle$ is an inner product. It induces the operator norm because $\langle \alpha_x, \alpha_x \rangle = \langle x, x \rangle = \|x\|^2 = \|\alpha_x\|^2$. Uniqueness comes from the polarization identity. For reflexivity, assume a continuous linear functional F on X^*. Since X^* is again a Hilbert space, by what we have proved so far it follows that there exists α_x such that $F(\alpha_y) = \langle \alpha_x, \alpha_y \rangle = \langle y, x \rangle = \alpha_y(x) = \widehat{x}(\alpha_y)$ for every $y \in X$. Since every $\beta \in X^*$ is of the form α_y it follows that $F = \widehat{x}$. ∎

The linear algebra argument in the parenthesis of the latter proof can be generalized to prove the following **linear nullstellensatz**: *If $\alpha_1, \ldots, \alpha_n, \beta$ are linear functionals on a vector space X then $\bigcap \mathrm{Ker}_{\alpha_j} \subseteq \mathrm{Ker}_\beta$ (namely β vanishes as soon as all α_j do) if and only if $\beta = \sum C_j \alpha_j$ for some scalars C_j.* *Proof.* To prove the only if part one can assume that α_j are linearly independent. Choose dual basis x_j, namely elements $x_1, \ldots, x_n$ in X such that $\alpha_i(x_j)$ equals the Kronecker delta δ_{ij}. For every $x \in X$ then $x - \sum \alpha_j(x)x_j \in \bigcap \mathrm{Ker}_{\alpha_j}$, hence $\beta = \sum \beta(x_j)\alpha_j$. Other proofs can be found in [Rud-FA, 3.9][HK, page 110]. Q.E.D.

Theorem 12. *For every bounded operator $T : X \to Y$ between Hilbert spaces there exists a unique bounded operator $T^* : Y \to X$, called the* **adjoint** *of T, such that $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for every $x \in X$ and $y \in Y$. Furthermore, $* : B(X;Y) \to B(Y;X)$, $T \mapsto T^*$, is a conjugate linear operation with extra properties*

$$T^{**} = T, \quad \|T^*\| = \|T\|, \quad \|TT^*\| = \|T\|^2.$$

Proof. There is only one way to construct T^*, as follows. Fix $y \in Y$. Consider the linear functional $X \to \mathbb{C}$, $x \mapsto \langle y, Tx \rangle$. Its norm is bounded by $\|T\|\|y\|$ according to the Cauchy-Schwarz inequality, so by the Riesz representation theorem, there exists a unique element $z \in X$ such that $\langle y, Tx \rangle = \langle z, x \rangle$, or equivalently, $\langle Tx, y \rangle = \langle x, z \rangle$. Set $T^*y = z$. For every $x \in X$ and $y \in Y$ we have

$$\langle y, T^{**}x \rangle = \langle T^*y, x \rangle = \overline{\langle x, T^*y \rangle} = \overline{\langle Tx, y \rangle} = \langle y, Tx \rangle,$$

hence $T^{**} = T$. Taking supremum over $\{x \in X : \|x\| \leq 1\}$ of the inequality

$$\|Tx\|^2 = |\langle x, T^*Tx\rangle| \leq \|T^*Tx\| \leq \|T^*\|\|Tx\| \leq \|T^*\|\|T\|\|x\|,$$

gives $\|T\|^2 \leq \|T^*T\| \leq \|T^*\|\|T\|$. Together with $T^{**} = T$ these inequalities give $\|T^*\| = \|T\|$ and $\|TT^*\| = \|T\|^2$. ■

Example 13. $S : l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $(a_0, a_1, a_2 \ldots) \mapsto (0, a_0, a_1, \ldots)$, is called the **(unilateral) forward shift**, one of the most important operators in functional analysis. Its adjoint $(a_0, a_1, a_2 \ldots) \mapsto (a_1, a_2, \ldots)$, is called the **(unilateral) backward shift**. ■

Exercise: Assume a Hilbert space X, and finitely many elements $x_1, \ldots, x_m$ of it. Show that the analysis and synthesis operators

$$X \to \mathbb{C}^m, \quad x \mapsto (\langle x, x_j\rangle), \qquad \mathbb{C}^m \to X, \quad (a_j) \mapsto \sum a_j x_j$$

are adjoint to each other.

Exercise: Show that an operator T on a Hilbert space X is the orthogonal projection onto closed linear subspace $Y \subseteq X$ if and only if $T \circ T = T = T^*$ and $Y = \mathrm{Ran}_T$. (*Hint.* $T \circ T = T$ implies $\mathrm{Ran}_T = \mathrm{Ker}_{1-T}$.)

Exercise: Let T be an operator on a Hilbert space X such that $\langle Tx, x\rangle = 0$ for every $x \in X$. Show that: (1) If $\mathbb{F} = \mathbb{C}$ then $T = 0$. (2) If $\mathbb{F} = \mathbb{R}$ and $T = T^*$ then $T = 0$. (3) $T = T^*$ can not be dropped in (2). (*Hint.* For (1), inspect $\langle Tz, z\rangle$ for $z = x \pm y$ and $z = x \pm \sqrt{-1}y$. For (3), think about rotations.)

Theorem 14. *Let $T : X \to Y$ be an operator between Hilbert spaces. Then:*

(1) T is an isometry (namely $\|Tx\| = \|x\|$ for every $x \in X$) if and only if it preserves the inner product (namely $\langle Tx, Ty\rangle = \langle x, y\rangle$ for every $x, y \in X$) if and only if $T^*T = 1$.

*(2) T is an isomorphism of Hilbert spaces (namely a bijective linear map which preserves the inner product) if and only if it is a surjective map which preserves the norm if and only if it is **unitary** in the sense that $T^*T = 1$ and $TT^* = 1$.*

(3; Mazur-Ulam) Every map between normed real vector spaces which preserves the distance and the origin is linear.

Proof. (1) $\|Tx\| = \|x\|$ for every $x \in X$ implies $\langle Tx, Ty\rangle = \langle x, y\rangle$ for every $x, y \in X$ via the polarization identity.

(2) Immediate from (1).

(3) The general case is proved in [Lax]. Here we treat the Hilbert space case. The only property of Hilbert spaces that we use is that: The triangle inequality $\|x+y\| \leq \|x\|+\|y\|$ is strict unless one of x or y is a nonnegative multiple of the other. (This is straightforward to check recalling the equality condition in the Cauchy-Schwarz inequality.) Let $T : X \to Y$ be such a map. Fix $x, y \in X$ and set $z := (x + y)/2$. Then

$$\begin{aligned}\|(Tx-Tz)+(Tz-Ty)\| = \|Tx-Ty\| = \|x-y\| = 2\|(x-y)/2\| = \|x-z\|+\|z-y\| = \\ = \|Tx - Tz\| + \|Tz - Ty\|,\end{aligned}$$

implies that one of $Tx-Tz$ or $Tz-Ty$ is a nonnegative multiple of the other. Since both has norm $\|x-y\|/2$ it follows that $Tx-Tz = Tz-Ty$, namely $Tx+Ty = 2T((x+y)/2)$. Putting $y = 0$ gives $Tx = 2T(x/2)$, hence $T(x+y) = Tx+Ty$. By induction $T(ax) = aTx$ for rational scalars a. By continuity $T(ax) = aTx$ for real scalars a. ■

Exercise: Why Mazur-Ulam theorem is not true for complex normed vector spaces?

Lemma 15 (Parseval Inequality)**.** *Let S be an orthonormal subset of a Hilbert space X in the sense that all elements of S are of norm one, and every two distinct elements are orthogonal to each other. Then $\sum_{s\in S}|\langle x,s\rangle|^2 \le \|x\|^2$ for every $x\in X$.*[3]

Proof. For every finite subset $F\subseteq S$ we have:

$$\left\|x-\sum_{s\in F}\overline{\langle x,s\rangle}s\right\|^2 = \|x\|^2 - 2\mathrm{Re}\left\langle x, \sum_{s\in F}\overline{\langle s,x\rangle}s\right\rangle + \sum_{s\in F}|\langle x,s\rangle|^2 = \|x\|^2 - \sum_{s\in F}|\langle x,s\rangle|^2.$$

■

Theorem 16 (Orthonormal basis)**.** *For an orthonormal subset S of a Hilbert space X the followings are equivalent:*

(1) S is maximal among orthonormal sets, or equivalently $S^\perp=\{0\}$.

(2) $x=\sum_{s\in S}\overline{\langle x,s\rangle}s$ for every $x\in X$. (Note that by Lemma 15 the summation on the right hand side has only countably many nonzero terms. The meaning of the convergnce of this series is that the series converges in the topology of X no matter how its terms are enumerated.)

(3) $\|x\|^2=\sum_{s\in S}|\langle x,s\rangle|^2$ for every $x\in X$. This is called the **Parseval identity***. Equivalently, $\langle x,y\rangle = \sum_{s\in S}\overline{\langle x,s\rangle}\langle y,s\rangle$ for every $x,y\in X$.*

(4) $X\to l^2(S)$, $x\mapsto(\langle s,x\rangle)_{s\in S}$, is an isomorphism of Hilbert spaces.

(5) The closed linear span of S is X. (Closed linear span is defined in page 25)

If any of these conditions hold then S is called an ***orthonormal basis*** *for X.*

Proof. (1)⇒(2) Let s_j, $j\in\mathbb{N}$, be an enumeration of all those $s\in S$ such that $\langle x,s\rangle\neq 0$. The series $\sum\overline{\langle x,s_j\rangle}s_j$ is Cauchy because $\sum|\langle x,s_j\rangle|^2$ is so:

$$\left\|\sum_{j=0}^{m}\overline{\langle x,s_j\rangle}s_j - \sum_{j=0}^{n}\overline{\langle x,s_j\rangle}s_j\right\|^2 = \sum_{j=0}^{m}|\langle x,s_j\rangle|^2 - \sum_{j=0}^{n}|\langle x,s_j\rangle|^2, \quad \forall m>n.$$

Let $y\in X$ be the convergence point of the series. By continuity, $\langle y,s\rangle=\langle x,s\rangle$ for all $s\in S$, so according to (1) we have $y=x$.

(2)⇒(3) Immediate from the continuity of the norm.

(3)⇒(1) Trivial.

(3)⇒(4) (3) says that the map is an isometry. To prove surjectivity assume $(c_s)_{s\in S}\in l^2(S)$. Let s_j be an enumeration of the countable set of all those $s\in S$ such that $c_s\neq 0$. Then the arguments in (2) shows that $\sum c_{s_j}s_j$ converges and the convergence point is a preimage of $(c_s)_{s\in S}$. *Another argument for surjectivity.* The range of the map in the

[3]Here is the definition and some elementary facts about the arbitrary sums of nonnegative numbers. If $(x_\alpha)_{\alpha\in A}$ is a collection of nonnegative real numbers indexed over a set A then $\sum x_\alpha$ is defined to be the supremum of all $\sum_{\alpha\in F}x_\alpha$, F varying over finite subsets of A. Let B be the set of all $\alpha\in A$ such that $x_\alpha>0$. One can easily show that if B is uncountable then $\sum x_\alpha=\infty$, but if $B=\{\beta_1,\beta_2,\ldots\}$ is countable then $\sum x_\alpha$ equals the usual series $\sum_{j=1}^{\infty}x_{\beta_j}$ namely $\lim_{k\to\infty}\sum_{j=1}^{k}x_{\beta_j}$. Refer [Fol, page 11] for details.

statement of (4) is dense and closed. (A standard Cauchy sequences argument shows that the range of any isometry is closed.)

(4)⇒(3) The map is an isometry.

(2)⇒(5) Trivial.

(5)⇒(1) Assume $x \in S^{\perp}$. Given $\epsilon > 0$, find a finite linear combination y of elements of S such that $\|x - y\| < \epsilon$. Since x is orthogonal to y, it follows that $\|x\|^2 \leq \|x\|^2 + \|y\|^2 = \|x - y\|^2 < \epsilon^2$. Therefore, $x = 0$. ■

Exercise: Prove that unitary operators are exactly the operators of the change of bases, namely, an operator T on a Hilbert space X is unitary if and only if there exist orthonormal bases $\{x_\alpha : \alpha \in A\}$ and $\{y_\alpha : \alpha \in A\}$ for X such that $Tx_\alpha = y_\alpha$ for all α.

There is an alternative way, based on the notion of nets [Fol, section 4.3], to make sense of the series like the one in the right hand side of the equation in Theorem 16.(2). Here are the definitions. A poset $(A, \leq)$ is called a **directed set** if for every $\alpha, \beta \in A$ there exists $\gamma \in A$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. A **net** $(x_\alpha)_{\alpha \in A}$ in a set X is a collection of elements $x_\alpha \in X$ indexed over a directed set A, namely a function $A \to X$. Now let X be a normed vector space. One says that the net (x_α) **converges** to $x \in X$ if for every $\epsilon > 0$ there exists $\alpha_0 \in A$ such that $\|x_\alpha - x\| < \epsilon$ for every $\alpha \geq \alpha_0$. It is straightforward to show that X is Banach if and only if every **Cauchy** net (namely for every $\epsilon > 0$ there exists $\alpha \in A$ such that $\|x_\beta - x_\gamma\| < \epsilon$ for every $\beta, \gamma \geq \alpha$) converges [Dou, page 3]. One says that the series $\sum_{\alpha \in A} x_\alpha$ **converses** to $x \in X$ if the net $y_F = \sum_{\alpha \in F} x_\alpha$ indexed over the directed set $(\{F : F \subseteq A, F \text{ finite}\}, \subseteq)$ of all finite subsets of A ordered by inclusion, converges to x. Here is a basic fact:

Theorem 17. *If x_α is a collection of orthogonal points in a Hilbert space X then $\sum x_\alpha$ converges if and only if $\sum \|x_\alpha\|^2$ converges. If so then there is only countably many nonzero x_α, $\sum x_\alpha$ converges to the usual series of any enumeration of α's such that $x_\alpha \neq 0$, and we have $\|\sum x_\alpha\|^2 = \sum \|x_\alpha\|^2$.*

Proof. Only if part is immediate from the continuity of the inner product. Conversely, assume that $\sum \|x_\alpha\|^2 < \infty$. Therefore for every $\epsilon > 0$ there exists finite subset $F_0 \subseteq A$ such that $\sum_{\alpha \in F} \|x_\alpha\|^2 - \sum_{\alpha \in F_0} \|x_\alpha\|^2 < \epsilon$ for every finite subset $F \subseteq A$ which contains F_0. For every finite subsets $F_1, F_2 \subseteq A$ which contain F_0 we have

$$\left\| \sum_{\alpha \in F_2} x_\alpha - \sum_{\alpha \in F_1} x_\alpha \right\|^2 \leq \sum_{\alpha \in F_1 \cup F_2} \|x_\alpha\|^2 - \sum_{\alpha \in F_0} \|x_\alpha\|^2 < \epsilon.$$

This shows that $\sum x_\alpha$ is Cauchy, hence convergent [Dou, page 3]. The rest follows by applying Theorem 16 to the Hilbert space Y of the closed linear span of $\{x_\alpha\}$. ■

Exercise: Let $(x_j)_{j \in \mathbb{N}}$ be a sequence in a normed vector space X. Show that: (1) If $\sum_{j \in \mathbb{N}} x_j$ converges x (in the meaning introduced on page 24) then $\lim_{k \to \infty} \sum_{j=1}^{k} x_j$ also converges x. (2) If $\lim_{k \to \infty} \sum_{j=1}^{k} \|x_j\|$ converges then $\sum_{j \in \mathbb{N}} x_j$ converges (in the meaning introduced on page 24). (3) If $X = \mathbb{C}$, then $\sum_{j \in \mathbb{N}} x_j$ converges if and only if $x_1 + x_2 + \cdots$ and any of its rearrangements converge. (The latter is equivalent to the convergence of $|x_1| + |x_2| + \cdots$ [Apo-A, 8.32-3].)

The **closed linear span** $\overline{\text{span}}(S)$ of a subset S of a normed vector space X is the smallest closed linear subspace of X containing S. It is the closure of the set of all linear combinations of elements of S. Here is a very useful theorem:

Theorem 18 (Closed linear span)**.** *Let S be a subset of a Hilbert space X.*

(1; Hahn-Banach density theorem) The closed linear span of S is the whole space X if and only if $S^\perp = \{0\}$. More generally, the closed linear span of S is given by $S^{\perp\perp}$.

(2) If S is orthonormal then the closed linear span of S is the set of all $\sum c_\alpha s_\alpha$ which $c_\alpha \in \mathbb{C}$, $s_\alpha \in S$ and $\sum |c_\alpha|^2 < \infty$.

Proof. (1) Set $Y := \overline{\text{span}}(S)$. We start with the first statement. Only if part is trivial. For the if part assume $x \in X$, and let $x = y + z$, $y \in Y$, $z \in Y^\perp$, be its orthogonal decomposition. Since $Y^\perp = S^\perp = \{0\}$, it follows that $z = 0$, hence $x = y \in Y$. Now we prove the second statement. Clearly, $S^{\perp\perp}$ is a closed linear subspace containing S, so $S^{\perp\perp} \supseteq Y$. For the other containment fix $x \in S^{\perp\perp}$, and let $x = y + z$, $y \in Y$, $z \in Y^\perp$, be its orthogonal decomposition. We have $\|z\|^2 = \langle z, x - y\rangle = 0 - 0 = 0$, so $z = 0$, hence $x = y \in Y$.

(2) Let Y be the closed linear span of S. Then Y is a Hilbert space with orthonormal basis S (Theorem 16.(5)), so the statement follows from Theorem 16.(4). Refer [Dou, page 68] for a direct proof. ∎

Theorem 19. *For every bounded operator $T : X \to Y$ between Hilbert spaces we have:*

(1)

$$\text{Ker}_T = \text{Ran}_{T^*}^\perp, \quad \overline{\text{Ran}_T} = \text{Ker}_{T^*}^\perp.$$

(2)

$$\|T\| = \sup\{|\langle Tx, y\rangle| : x, y \in X, \|x\| \le 1, \|y\| \le 1\}.$$

Proof. (1) The first identity is immediate from the definitions:

$$x \in \text{Ker}_T \leftrightarrow Tx = 0 \leftrightarrow \langle Tx, y\rangle = 0, \forall y \in Y \leftrightarrow \langle x, T^*y\rangle = 0, \forall y \in Y \leftrightarrow x \in \text{Ran}_{T^*}^\perp.$$

Replacing T by T^* in the first identity, taking orthogonal complement, and using Theorem 18.(1) gives the second identity.

(2) We have

$$\|T\| = \sup_{\|x\|\le 1} \|Tx\| = \sup_{\|x\|\le 1} \sup_{\|y\|\le 1} |\langle Tx, y\rangle|,$$

by the isometry part in the Riesz representation theorem. ∎

Exercise: Let X be a Hilbert space and $Y \subseteq X$ be a closed linear subspace with orthonormal basis $\{y_\alpha\}$. Show that the orthogonal projection P in X onto Y is given by $Px = \sum \langle y_\alpha, x\rangle y_\alpha$ for every $x \in X$.

Example 20. Here are some famous orthonormal bases:

1. Let S be a set. Given $s \in S$, let $e_s : S \to \mathbb{C}$ be the function given by $e_s(t) = 1$ if $t = s$ and 0 otherwise. Then the functions e_s, $s \in S$, constitute an orthonormal basis for $l^2(S)$.

2. The functions $\sqrt{\frac{j+1}{\pi}}z^j$, $j \in \mathbb{N}$, constitute an orthonormal basis for the Bergman space $L^2_a(\mathbb{D})$ on the unit disc of the complex plane.

   (*Proof.* Orthonormality is clear by a computation in polar coordinates:

$$\int_{\mathbb{D}} \overline{z}^j z^k d\mu(z) = \int_0^1 \int_0^{2\pi} r^{j+k+1} e^{\sqrt{-1}2\pi(-j+k)} d\theta dr = \frac{\pi}{j+1}\delta_{jk},$$

   where δ is the Kronecker tensor. Given $f \in L^2_a$, consider its shrinked version $f_r(z) = f(rz)$, $z \in \mathbb{D}$, $r \in (0,1)$. Each f_r is a continuous function on the closure of $\mathbb{D}$, so according to the Stone-Weierstrass theorem (Application 51), can be sufficiently approximated by linear combinations of the basis elements in the uniform topology, hence in the L^2 topology as well. It remains to show that $f_r \to f$ in L^2 as $r \to 1$. Decompose the integral $\int_{\mathbb{D}} |f - f_r|^2 d\mu(z)$ into the one over $|z| \leq 1-\epsilon$ and over $1-\epsilon < |z| < 1$, where $\epsilon \in (0,1)$. The first integral approaches 0 as $r \to 1-$ and for each ϵ, because of the uniform continuity of f on $|z| \leq 1-\epsilon$. The second integral approaches 0 as $\epsilon \to 1$ by the Lebesgue dominated convergence theorem. The whole argument shows that L^2 is the closed linear span of our basis. (Exactly the same argument shows that polynomials are dense in $L^p_a(\mathbb{B}_m)$, $p \in [1,\infty]$, where $\mathbb{B}_m$ is the unit open ball of $\mathbb{C}^m$ [Zhu-FT, page 43].) *A Hilbert spaces argument [DS, page 11].* Fix $f = \sum a_j z^j \in L^2_a$, with the uniform convergence on compacts subsets of $\mathbb{D}$. We need to verify that the same representation hold in L^2 topology. Let s_j be the partial sums of $\sum a_j z^j$. Given $r \in (0,1)$, we have

$$\int_{|z|\leq r} |s_j(z)|^2 d\mu(z) = \sum_{k=0}^{j} \frac{\pi}{k+1}|a_k|^2 r^{2k+2}.$$

   Since $s_j \to f$ uniformly on $|z| \leq r$, it follows that

$$\int_{|z|\leq r} |f(z)|^2 d\mu(z) = \sum_{k=0}^{\infty} \frac{\pi}{k+1}|a_k|^2 r^{2k+2}.$$

   By the Lebesgue dominated convergence theorem, $\|f\|^2 = \sum_{k=0}^{\infty} \frac{\pi}{k+1}|a_k|^2$. With the same arguments one can show that $\langle f, z^k \rangle = \frac{\pi}{k+1}\overline{a_k}$. We have proved the Parseval identity $\|f\|^2 = \sum |\langle f, e_k \rangle|^2$, $e_k := \sqrt{(k+1)/\pi}z^k$.)

3. The functions $\exp(\sqrt{-1}2\pi jx)$, $j \in \mathbb{Z}$, constitute an orthonormal basis for $L^2([0,1], dx)$, called the **Fourier basis** [Fol, 8.20]. Here dx denotes the Lebesgue measure.

   (*Proof.* Orthonormality is clear. Each element of L^2 can be sufficiently approximated by continuous function. By another approximation, these continuous functions can be assumed to have equal values at the endpoints 0 and 1. These latter continuous functions can be thought to live on the unit circle $\mathbb{S}^1 = \{\exp(\sqrt{-1}2\pi x) : x \in [0,1]\}$, a compact space. By the Stone-Weierstrass theorem the finite linear combinations of our basis elements are dense in $C(\mathbb{S}^1)$ in the uniform topology, hence in the L^2 topology. The whole argument shows that L^2 is the closed linear span of our basis. *Remark.* Instead of the Stone-Weierstrass theorem, one can use the Cesaro summability of the Fejer means [Apo-A, 11.15, 11.17].)

4. The functions $\exp(\sqrt{-1}2\pi jx)1_{[0,1]}(x-k)$, $j, k \in \mathbb{Z}$, constitute an orthonormal basis for $L^2(\mathbb{R}, dx)$, sometimes called the **Gabor basis** [Dau, page 108].

5. Consider the functions $\psi : [0,1] \to \mathbb{R}$ given by

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \end{cases}.$$

   Then the constant function 1 together with $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, $j \in \mathbb{N}$, $k = 0, \dots, 2^j - 1$, constitute an orthonormal basis for $L^2([0,1], dx)$.

   (*Proof.* Orthonormality is clear. The key observation is that the characteristic function of every dyadic interval $[m2^{-n}, (m+1)2^{-n})$, $n \in \mathbb{N}$, $m = 0, 1, \dots, 2^n - 1$, is a finite linear combination of basis elements. On the other hand, $L^2([0,1])$ is the closed linear span of dyadic characteristic functions (by the proof of [Fol, Theorem 2.10]).)

6. Consider the function $\psi : \mathbb{R} \to \mathbb{R}$ given by

$$\psi(x) = \begin{cases} 1, & 0 \le x < \frac{1}{2} \\ -1, & \frac{1}{2} \le x < 1 \\ 0, & \text{otherwise} \end{cases}. \tag{2.3}$$

   Then the functions $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, constitute an orthonormal basis for $L^2(\mathbb{R}, dx)$, sometimes called the **Haar basis** [Dau, pages 10–13]. Any $\psi \in L^2(\mathbb{R}, dx)$ such that its dilated translations $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, $j, k \in \mathbb{Z}$, constitute an orthonormal basis for $L^2(\mathbb{R}, dx)$ is called a **wavelet**. There exists smooth wavelet functions in the literature. Wavelet theory is a rich realm of pure mathematics with a lot of applications. A good start is [Dau].

7. In the Hilbert space $L^2([0,1], dx)$, the monomials $1, x, x^2, \dots$ constitute a countable dense sequence according to the Stone-Weierstrass theorem. Applying the Gram-Schmidt process to this sequence gives an orthonormal basis, according to the proof of Theorem 22. The j-th element of this orthonormal basis is given by $d^j(x^2-2)^j/dx^j$, up to a constant C_j, and is called the Legendre polynomial of degree j. ■

Exercise: Consider the generator ψ of the Haar wavelet (2.3), and let χ be the characteristic function of the interval $[0,1)$. Prove that $\chi(x-k)$, $k \in \mathbb{Z}$, together with $\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k)$, $j \in \mathbb{N}$, $k \in \mathbb{Z}$, constitute an orthonormal basis for $L^2(\mathbb{R}, dx)$. (*Hint.* Imitate the proof in Example 20, part 5.)

Theorem 21 (Structure theorem of Hilbert spaces). *Every Hilbert space has an orthonormal basis, and any two orthonormal bases have the same cardinality, called the* ***dimension*** *of that Hilbert space. Furthermore, two Hilbert spaces are isomorphic as Hilbert spaces if and only if they have the same dimension.*

$$\boxed{\textit{Every Hilbert space is an } l^2(X) \textit{ space.}}$$

Proof. Consider the poset of all orthonormal subsets of a Hilbert space X, ordered by inclusion. Every chain in this poset has an upper bound: the union of the elements of the chain. By Zorn lemma this poset has a maximal element, which is an orthonormal basis by Theorem 16.(1). Next, assuming two orthonormal bases S and T for X we want to show that they have the same cardinality. The result follows from linear algebra if at least one of S or T is finite [HK, page 44], so assume both are infinite. To any $s \in S$ one can assign the nonempty countable collection of those points $t \in T$ such that $\langle s, t\rangle \neq 0$. (Nonemptyness is because T is an orthonormal basis and $s \neq 0$.) Any element of T belongs to at least one of the collections in the range this assignment (because S is an orthonormal basis), hence $|T| \leq |S||\mathbb{N}| = |S|$. Similarly $|S| \leq |T|$. By the Cantor-Bernstein theorem we have $|S| = |T|$. The rest is straightforward. ∎

Exercise: Show that every orthonormal set in a Hilbert space is contained in some orthonormal basis. (*Hint.* Apply Zorn lemma.)

Theorem 22. *A Hilbert space is separable if and only if it has countable orthonormal basis (equivalently, its dimension is countable).*

Proof. If S is a countable orthonormal basis for a Hilbert space X then the finite linear combinations of elements of S with coefficients in $\mathbb{Q} + \sqrt{-1}\mathbb{Q} \subseteq \mathbb{C}$ is dense in X. Conversely, assume a Hilbert space X with a countable dense sequence x_j in it. Inductively discarding those x_j which belong to the linear span of $x_1, \ldots, x_{j-1}$, and then refreshing indices one can assume $x_1, x_2, \ldots$ to be linearly independent (in the sense of linear algebra). Apply the Gram-Schmidt process

$$y_1 = x_1, \quad y_j = x_j - \sum_{k=1}^{j-1} \frac{\langle x_j, x_k\rangle}{\langle x_k, x_k\rangle} x_k, \quad j = 2, 3, \ldots,$$

to make $\{y_1, y_2, \ldots\}$ an orthogonal set. Since the linear span of $x_1, \ldots, x_j$ equals the linear span of $y_1, \ldots, y_j$ for each j it follows that the closed linear span of $y_1, y_2, \ldots$ is the whole X. Replacing y_j by $y_j/\|y_j\|$ gives an orthonormal basis for X. *Another argument.* If S is an orthonormal basis of a Hilbert space X then the open balls of radius $1/\sqrt{2}$ around elements of S are pairwise disjoint, so if S is uncountable then X can not have a countable dense subset. ∎

Putting Theorem 16 and Example 20.(2) together we have:

Application 23 (Riesz-Fischer-Parseval)**.** *The Fourier series of an $L^2([0,1], dx)$ function f converges f in L^2-norm namely $\int_0^1 |f(x) - S_j(x)|^2 dx \to 0$ as $j \to \infty$, where $S_j(x) = \sum_{|k| \leq j} c_k \exp(2\pi\sqrt{-1}kx)$ and $c_k = \int_0^1 f(x) \exp(-2\pi\sqrt{-1}kx)dx$, $k \in \mathbb{Z}$. Furthermore, the Parseval identity $\int_0^1 |f(x)|^2 dx = \sum |c_k|^2$ holds. Conversely, for any sequence $(c_k)_{k\in\mathbb{Z}}$ of complex numbers which $\sum |c_k|^2 < \infty$ there exists $f \in L^2([0,1], dx)$ such that $c_k = \int_0^1 f(x) \exp(-2\pi\sqrt{-1}kx)dx$.*

Exercise: Let a_j be a sequence of real numbers such that $\sum a_j b_j < \infty$ for every sequence of real numbers b_j which $\sum b_j^2 < \infty$. Prove that $\sum a_j^2 < \infty$. (*Hint.* Use the Banach-Steinhaus theorem along with the Riesz representation theorem.)

Exercise: A sequence x_j in a Hilbert space X is said to **weakly converge** $x \in X$ if $\langle x_j, y\rangle \to \langle x, y\rangle$ for every $y \in X$. Prove that a linear map $T : X \to Y$ between Hilbert spaces is continuous if and only if for every sequence x_j in X which weakly converges x it is the case that Tx_j weakly converges Tx. (*Hint.* Use the closed graph theorem.)

Chapter 3

Spaces of functions II: Topological vector spaces

References: [Rud-FA, chapter 1].

In some areas of mathematrical analysis functions spaces appear whose topology can not be induced by a single norm, or they are not even metrizable. Functional analysts developed a beautiful theory for a very general class of function spaces applicable in such situations: the theory of topological vector spaces. This is the subject of this chapter. Even a special class of these spaces (called locally convex topological vector spaces) allows a rich duality theory. Here is a second reason to study such spaces. In the study of normed vector spaces there are topologies (different from the one induced by the norm) which are very useful but are rarely metrizable (Theorem 41, Example 43); these are weak and weak-star topologies to be discussed in Chapter 5. The good news is that these topologies are locally convex most of the time (Theorem 34). One last emphasize: Topological vector spaces are important because on one hand they are so general to cover most spaces appearing in mathematical analysis, and on the other hand they allow a rich theory (a duality theory (Theorem 52) and the fundamental theorems of Banach space theory (Remark 24).)

Let $\mathbb{F}$ be either the field of reals or the field of complex numbers. A **topological vector space** (**TVS** for short) X is a $\mathbb{F}$-vector space equipped with a T_1 topology (namely, singletons are closed) such that the scalar multiplication and addition operations of vector spaces are continuous.[1] For example, all normed vector spaces are such. The most important elementary fact about TVSs is that the translation by a fixed vector as well the multiplication by a nonzero scalar are homeomorphisms.

3.1 Elementary theory of topological vector spaces

Here are some fundamental definitions and facts of the theory. Let X be a topological vector space, and $S \subseteq X$ a subset.

[1]Some references drop T_1 separation axiom. We are following Rudin [Rud-FA]. Hausdorff axiom (T_2) is then deduced.

1. A **local basis** (of the origin) for X is a collection of open neighborhoods of the origin such that any other neighborhood of the origin contains at least one member of this family. Therefore every open subset of the topology is a union of translations of some members of a local basis.

2. S is called **balanced** if $aS \subseteq S$ for every $a \in \mathbb{F}$ with $|a| \leq 1$.

   Exercise: Describe all balanced subsets of $\mathbb{R}^2$ and $\mathbb{C}$ as TVSs over $\mathbb{R}$ and $\mathbb{C}$, respectively.

3. Every neighborhood U of the origin of X contains a balanced neighborhood V of the origin such that $V + V \subseteq U$, or even $V + V + V + V \subseteq U$. Every convex neighborhood U of the origin of X contains a balanced, convex neighborhood V of the origin.

   (*Proof.* Since the addition of vectors is continuous at the point $0 + 0 = 0$ there exist neighborhoods V_1, V_2 of 0 such $V_1 + V_2 \subseteq U$. By the continuity of the scalar multiplication in X there exists a neighborhood V_1' of 0 such that $aV_1' \subseteq V_1$ for every $|a| \leq \epsilon$ and some $\epsilon > 0$. Then $V_1'' := \bigcup_{|a|\leq\epsilon} aV_1'$ is a balanced neighborhood of 0 contained in V_1. Similarly construct V_2''. Then $V := V_1'' \cap V_2''$ is a balanced neighborhood of 0 such that $V + V \subseteq U$. To get $V + V + V + V \subseteq U$ repeat the process. Now assume that U is convex. Set $A := \bigcap_{|a|=1} aU$ and let V be a balanced neighborhood of the origin such that $V \subseteq U$. Since V is balanced if follows that $V \subseteq A$, so the interior $\mathrm{int}(A)$ of A is a neighborhood of 0. Being an intersection of convex sets, A is convex, hence so is $\mathrm{int}(A)$. It remains to show that $\mathrm{int}(A)$ is balanced. For every $0 < r < 1$ and $b \in \mathbb{F}$ of modulus 1 we have $rbA = \bigcap_{|a|=1} rbaU = \bigcap_{|a|=1} raU$. Since aU is a convex set containing 0 it also contains raU. Therefore $rbA \subseteq A$.)

4. X has a balanced local basis. If X is locally convex (to be defined in (17)) then X has a balanced convex local basis.

   (*Proof.* Immediate from (3).)

5. X is Hausdorff.

   (*Proof.* Assume a nonzero element $x \in X$. Since 0 belongs to the open subset $X \setminus \{x\} \subseteq X$, there exists a neighborhood U of 0 which does not contain x. Find a balanced neighborhood V of 0 such that $V + V \subseteq U$. Then V and $x + V$ are disjoint neighborhoods of 0 and x, respectively.)

6. Here is a stronger separation result that will be needed for proving the most important result of this course (Theorem 25.(4)): For every compact $K \subseteq X$ and closed $C \subseteq X$ which are disjoint one can find neighborhood U of the origin such that $(K + U) \cap (C + U) = \emptyset$.

   (*Proof.* For every $x \in K$, since $x \notin C$, one can find balanced neighborhood U_x of 0 such that $(x + U_x + U_x + U_x + U_x) \cap C = \emptyset$, which implies $(x + U_x + U_x + U_x) \cap (C + U_x) = \emptyset$. Compact K can be covered by finitely many $x_j + U_{x_j}$. Then $U := \bigcap U_{x_j}$ works because $K + U \subseteq \bigcup x_j + U_{x_j} + U \subseteq \bigcup x_j + U_{x_j} + U_{x_j}$.)

7. The closure of S equals the intersection of all $S + U$ where U ranges over a local basis. The closure of a linear subspace is a closed linear subspace.

(*Proof.* If $x \in \overline{S}$ then since $x - U$ is a neighborhood of x it follows that $x - U$ intersects S, so $x \in S + U$. Conversely, if $x \in \bigcap S + U$, since for every neighborhood V of 0 there exists some U such that $U \subseteq -V$ it follows that $x \in S + U \subseteq S - V$, namely $x + V$ intersects S. For the second statement let $Y \subseteq X$ be a linear subspace of X. Assume $x \in \overline{Y}$. For every neighborhood U of the origin we have $x + u = y$ for some $u \in U$ and $y \in Y$, so $ax + au = ay$ for every $a \in \mathbb{F}$, hence $ax \in \overline{Y}$. Assume $x_1, x_2 \in \overline{Y}$. For every neighborhood V of $x_1 + x_2$, by the continuity of the addition of vectors, there exists neighborhoods U_1 of x_1 and U_2 of x_2 such that $U_1 + U_2 \subseteq V$. Y intersects both U_1 and U_2, so it intersects U.)

8. S is called a **bounded** if for every neighborhood V of the origin there exists $r > 0$ such that $S \subseteq sV$ for every $s > r$. Since every neighborhood of the origin contains a balanced one it follows that an equivalent definition of boundedness is that for every neighborhood V of the origin there exists $r > 0$ such that $S \subseteq rV$. For a general metrizable space X this notion of boundedness is *not* equivalent with the famous one: There exists $C > 0$ such that $d(x, y) < C$ for every $x, y \in S$. (Refer [Rud-FA, page 23] for a counterexample.)

9. Let U be a neighborhood of the origin. Then $X = \bigcup_{j \in \mathbb{N}} jU$. If U is bounded then $\{j^{-1}U : j = 1, 2, \ldots\}$ is a local basis.

   (*Proof.* For every $x \in X$, by the continuity of the scalar multiplication at $0 \times x = 0$, there exists $\epsilon > 0$ such that $ax \in U$ whenever $|a| < \epsilon$. Therefore $x \in jU$ for every integer $j > 1/\epsilon$. For the second statement, assuming a neighborhood V of 0, there exists $r > 0$ such that $U \subseteq sV$ for every $s > r$. Therefore $j^{-1}U \subseteq V$ for every integer $j > r$.)

   Exercise: Why boundedness is needed in the latter statement?

10. A linear map $T : X \to Y$ between TVSs is called **bounded** if it maps bounded subsets to bounded ones. It is straightforward to check that continuity implies boundedness, but the converse is not true [Rud-FA, page 39, exercise 13]. If X is metrizable then T is bounded if and only if it is continuous if and only if $Tx_j \to 0$ for every sequence $x_j \to 0$ [Rud-FA, 1.32].

11. A linear functional $\alpha : X \to \mathbb{F}$ on X is continuous if and only if Ker_α is closed.

    (*Proof.* The only if part is due to $\mathrm{Ker}_\alpha = \alpha^{-1}(0)$. Conversely, putting the trivial case $\alpha \equiv 0$ aside, fix $x \in X \setminus \mathrm{Ker}_\alpha$. There exists a balanced neighborhood U of 0 such that α never vanishes on $x + U$. By linearity $\alpha(U) \subseteq \mathbb{F}$ is balanced. $\alpha(U) \neq \mathbb{F}$ because otherwise $\alpha(y) = -\alpha(x)$ for some $y \in U$, hence the contradiction $x + y \in \mathrm{Ker}_\alpha \cap x + U$. Therefore $\alpha(U)$ contains an open ball around 0. Continuity follows.)

12. Recall that a topological space is **metrizable** if its topology is induced by a metric. X is metrizable if and only if it has a countable local basis; and if so then the metric can be taken to be translation invariant, namely $d(x, y) = d(x + z, y + z)$ for every x, y, z [Rud-FA, 1.24].

13. S is called **absorbing** if for every $x \in X$ there exists $r > 0$ such that $x \in rS$.

Exercise: Give examples of an absorbing subset which is not balanced, and a balanced subset which is not absorbing.

14. Seminorms μ on a vector space X correspond (not necessarily in a one-to-to-one fashion) to convex absorbing balanced subsets $S \subseteq X$ via:

$$\mu \mapsto S_\mu := \{x \in X : \mu(x) < 1\},$$

$$S \mapsto \mu_S, \quad \mu_S(x) = \inf_{t>0}\{x \in tS\}.$$

μ_S is called the **Minkowski** (or **Gauge**) **functional** associated to S. The intuition about the gauge functional comes from the observation that if S is the unit ball of a normed vector space X then μ_S retrieves the norm. If X is a TVS then under the same maps, continuous seminorms are in one-to-one correspondence with convex balanced neighborhood of the origin, namely $S = S_{\mu_S}$ and $\mu = \mu_{S_\mu}$. In this situation $S = S_{\mu_S} = \{\mu_S < 1\}$ reveals the usefulness of the gauge functional: It makes convex balanced neighborhoods of the origin look like the open unit ball.

(*Proof.* All these statements are straightforward to check [Rud-FA, 1.34-6]. Let us just check that if S is a convex balanced neighborhood of the origin then $S = \{\mu_S < 1\}$. S is absorbing by (9). If $x \in S$ by openness of S there exists $t < 1$ such that $x \in tS$, so $\mu_S(x) < 1$. If $x \notin S$ then the set $\{t > 0 : x \in tS\}$ does not contain 1, so its infimum must by ≥ 1, because by convexity and absorbingness of S, for every $x \in X$ the set $\{t > 0 : x \in tS\}$ has the property that if it contains t then it contains $[t, \infty)$.)

15. Let $p_\alpha : X \to [0, \infty)$, $\alpha \in A$, be a family of seminorms on a vector space X. Then the weakest (namely smallest) topology such that all p_α are continuous is the one with subbasis $U_{\alpha,x,\epsilon} = \{y \in X : p_\alpha(x - y) < \epsilon\}$, $\alpha \in A$, $x \in X$, $\epsilon > 0$. Equivalently, a net $(x_i)_{i\in I}$ in X converges to x in this topology if and only if $p_\alpha(x_i - x) \to 0$ for every $\alpha \in A$. (Refer to page 48 for a review of the corresponding notions from general topology.) This family of seminorms is called **separating** if the topology it generates is Hausdorff. This happens if and only if there is no nonzero $x \in X$ such that all $p_\alpha(x)$ vanish.

16. Let $T : X \to Y$ be a linear map between TVSs X and Y whose topologies are induced, respectively, by the families of seminorms $\{p_\alpha\}_{\alpha\in A}$ and $\{p_\beta\}_{\beta\in B}$. Then T is continuous if and only if for every $\beta \in B$ there exists $C > 0$ and a finite subset $F \subseteq A$ such that $p_\beta(Tx) \leq C \sum_{\alpha\in F} p_\alpha(x)$ for every $x \in X$.

(*Proof.* Assume that T is continuous. For every $\beta \in B$, since $F^{-1}\{y \in Y : p_\beta(y) < 1\}$ is open and contains 0, it follows that there exists finitely many $\alpha_1, \ldots, \alpha_n$ in A and $\epsilon > 0$ such that $p_\beta(Tx) < 1$ whenever $p_{\alpha_j}(x) < \epsilon$ for all j. We assert that $C := 1/\epsilon$ works. Fix $x \in X$. If all $p_{\alpha_j}(x)$ vanish then all $p_{\alpha_j}(rx)$ vanish for every $r > 0$, hence $p_\beta(Tx) < 1/r$, which only happens if $p_\beta(Tx) = 0$. If at least one $p_\alpha(x)$ is nonzero then $p_{\alpha_j}(\epsilon x / \sum p_{\alpha_j}(x)) < \epsilon$ for all j, hence $p_\beta(\epsilon x / \sum p_{\alpha_j}(x)) < 1$. If part is trivial.)

17. X is called a **locally convex** (**LCTVS** for short) if it has a local basis whose members are convex. There is another equivalent definition which is the most common way these spaces appear in practice [Rud-FA, 27-29]: X is locally convex if and only

if its topology is induced by a separating family of seminorms p_α, $\alpha \in A$, on X, namely a local basis of the origin is given by finite intersections of sets of the form $U_{\alpha,\epsilon} := \{x \in X : p_\alpha(x) < \epsilon\}$, $\alpha \in A$, $\epsilon > 0$. (Of course $\epsilon > 0$ can be replaced by $1/j$, $j = 1, 2, \ldots$.) If part is easy to verify. Specially, T_1 axiom comes from "separating", and local convexity is because: If $x, y \in U_{\alpha,\epsilon}$ then for every $\lambda \in (0,1)$ we have $p_\alpha(\lambda x+(1-\lambda)y) \leq \lambda p_\alpha(x)+(1-\lambda)p_\alpha(y) < \epsilon$, hence $\lambda x+(1-\lambda)y \in U_{\alpha,\epsilon}$. For only if part, assuming a convex local basis for X, first construct a balanced convex local basis $\{U_\alpha\}_{\alpha \in A}$ by (4); then the gauge functionals μ_{U_α}, $\alpha \in A$, constitute a separating family of seminorms which induces the topology of X [Rud-FA, pages 27-29]. Here is a useful fact: If the topology of X is given by a separating family of seminorms p_α, $\alpha \in A$, then a subset $E \subseteq X$ is bounded if and only of each p_α is bounded on E in the sense that there exist $M_\alpha > 0$ such that $p_\alpha(x) < M_\alpha$ for every $x \in E$ and every $\alpha \in A$. If p_j, $j \in \mathbb{N}$, is a countable separating family of seminorms then $d(x,y) = \max \frac{2^{-j} p_j(x-y)}{1+p_j(x-y)}$ is a translation-invariant metric which induces the same topology [Rud-FA, 1.38.(c)].

$$\boxed{\text{LCTVSs} \leftrightarrow \text{separating families of seminorms}},$$

$$\boxed{\text{metrizable LCTVSs} \leftrightarrow \text{countable separating families of seminorms}}.$$

18. X is called **locally bounded** if it has a neighborhood of the origin which is bounded.

19. X is called **normable** if its topology is induced by a norm. This happens if and only if X is locally bounded and locally convex [Rud-FA, 1.39].

20. X is called **complete** if every Cauchy net converges. (A net $(x_\alpha)_{\alpha \in A}$ in X is called **Cauchy** if for every neighborhood U of the origin there exists $\alpha_0 \in A$ such that $x_\alpha - x_\beta \in U$ for every $\alpha, \beta \geq \alpha_0$.) If X is first-countable then X is complete if every Cauchy sequence converges.

21. X is called an **F-space** if the topology is complete and given by a translation-invariant metric.

22. X is called a **Frechet space** if it is a locally convex F-space. Equivalently, the topology of X is complete and given by a countable separating family of seminorms.

23. Every compact subset $A \subseteq X$ is closed and bounded. (*Proof.* Closedness is because X is Hausdorff [Mun, 26.3]. For boundedness assume a neighborhood U of 0. Since every neighborhood of 0 contains a smaller balanced one we can assume U is balanced. By (9), $A \subseteq \bigcup_{j \in \mathbb{N}} jU$. The compactness of A and balance of U implies that $A \subseteq jU$ for some j.) X is said to have the **Heine-Borel property** if every closed and bounded subset is compact.

24. Let Y a closed linear subspace of X. Then the vector space $X/Y := \{x + Y : x \in X\}$ of cosets of Y equipped with the quotient topology turn out to be a TVS, called the **quotient space**, and the **quotient map** $\pi : X \to X/Y$, $x \mapsto x + Y$, is a continuous linear map. (Interpret $x + Y$ as the set $\{x + y : y \in Y\}$, therefore two cosets $x + Y$ and $x' + Y$ are equal if and only it $x - x' \in Y$. Vector space operations are defined

naturally: $(x+Y)+(x'+Y)=(x+x')+Y$ and $a(x+Y)=ax+Y$, for every $x,x'\in X$ and $a\in\mathbb{F}$. X/Y is topologized by declaring $A\subseteq X/Y$ to be open if and only if $\pi^{-1}(A)$ is open is X.) Note that X/Y is satisfies T_1 axiom because the preimage under π of a singleton subset $\{x+Y\}$ of X/Y, being equal to $x+Y$, is closed in X. Here are some useful facts with straightforward proofs [Rud-FA, 1.41]:

- π is open, namely maps open subsets to open ones.
- If $\{U_\alpha\}$ is a local basis for X then $\{\pi(U_\alpha)\}$ is a local basis for X/Y.
- If X satisfies any of the following properties then X/Y satisfies the same: local convexity, local boundedness, metrizability, normability, F-space, Frechet space, Banach space.

Some remarks about the proof:

- If X is normable by $\|-\|_X$ then X/Y is normable by $\|x+Y\|=\inf\{\|x-y\|_X : y\in Y\}$, the distance of x to Y.
- If X is metrizable by invariant metric d_X then X/Y is metrizable by invariant metric $d(x+Y,x'+Y)=\inf\{d_X(x-x',y):y\in Y\}$.

25. If X is locally bounded then it has a countable local basis (equivalently, metrizable).

    If X is locally bounded and has the Heine-Borel property then it is locally compact.

    (*Proof.* Let U be a bounded neighborhood of 0. Then $j^{-1}U$, $j=1,2,\ldots$, is a countable local basis. For the second statement it suffices to show that $\overline{U}$ is bounded, because then by the Heine-Borel property $\overline{U}$ is also compact. Let V be an arbitrary neighborhood of 0. Since U is bounded it follows that there exists $r>0$ such that $U\subseteq sV$ if $s>r$. Assuming $x\in\overline{U}$, since $x-V$ is a neighborhood of x it follows that $x-v=u=rv'$ for some $v,v'\in V$, $u\in U$, hence $x=v+sv'$. This shows that $\overline{U}\subseteq(s+1)V$.)

26. If X has finite vector space dimension then there is exactly one topology on X which makes it a TVS.

    (*Proof.* Assume a vector space isomorphism $F:\mathbb{F}^m\to X$. With respect to a basis $(e_1,\ldots,e_m)$ for $\mathbb{F}^m$ the mapping F acts by $(a_1,\ldots,a_m)\mapsto\sum a_jF(e_j)$, hence continuous. Let B be the open unit ball of $\mathbb{F}^m$ and S be the boundary of $\overline{B}$. $K:=F(S)$ is compact and does not contain 0. By compactness there exists a (balanced) neighborhood V of 0 such that V and K do not intersect. Since F is bijective it follows that $U:=F^{-1}(V)$ and S do not intersect. Since F is linear if follows that U is balanced, hence connected. Therefore $U\subseteq B$ (otherwise U intersects S), which clearly implies that F^{-1} is continuous at the origin, hence everywhere by linearity.)

27. If Y and Z are, respectively, a closed and finite dimensional linear subspace of X then $Y+Z$ is closed.

    (*Proof.* First we reduce to $Y=0$ case. Consider the quotient $\pi:X\to X/Y$, and observe that $Y+Z=\pi^{-1}(\pi(Z))$. $\pi(Z)$ is finite dimensional because π being linear maps linear spanning sets of Z to linear spanning sets of $\pi(Z)$. Therefore we only need to verify that finite dimensional linear subspaces of TVSs are closed, so assume $Y = 0$. Fix a vector space isomorphism $F : \mathbb{F}^m \to Z$. With respect to a basis $(e_1, \dots, e_m)$ for $\mathbb{F}^m$ the mapping F acts by $(a_1, \dots, a_m) \mapsto \sum a_j F(e_j)$, hence continuous. Let B be the open unit ball of $\mathbb{F}^m$ and S be the boundary of $\overline{B}$. $K := F(S)$ is compact and does not contain 0. By compactness there exists a (balanced) neighborhood V of 0 in X such that V and K do not intersect. Since F is bijective it follows that $U := F^{-1}(V \cap Z)$ and S do not intersect. Since F is linear if follows that U is balanced, hence connected. Therefore $U \subseteq B$ (otherwise U intersects S), or equivalently $V \cap Z \subseteq F(B)$. Fix $p \in \overline{Z}$. Since $X = \bigcup_{j \in \mathbb{N}} jV$ there exists positive integer j such that p also belong to open $jV \subseteq X$. Therefore p belongs to the closure of $jV \cap Z \subseteq F(jB) \subseteq F(j\overline{B})$. However $F(j\overline{B})$ is compact, hence closed in X. Therefore $p \in F(j\overline{B}) \subseteq Z$.)

$$\boxed{\text{closed} + \text{finite dimensional} \to \text{closed}}.$$

28. Recall the notion of **local compactness** from general topology [Mun, Theorem 29.2]: X is locally compact if and only if it has a neighborhood of the origin whose closure is compact. This happens if and only if the vector space dimension of X is finite.

    (*Proof.* If X is of finite dimension n then by (26) X is homeomorphic to the Euclidean space $\mathbb{R}^n$, so the closed balls are compact by Heine-Borel theorem. Conversely, let U be a neighborhood of 0 in X such that $\overline{U}$ is compact. By compactness one can find finitely many points $x_1, \dots, x_n$ in X such that $\overline{U} \subseteq \bigcup x_j + \frac{1}{2}U$. Let Y be the linear span of $\{x_1, \dots, x_n\}$. Since $U \subseteq Y + \frac{1}{2}U$ and $Y = aY$ for every nonzero scalar a, it follows that $U \subseteq Y + \frac{1}{2}Y + \frac{1}{4}U = Y + \frac{1}{4}U$, so by repetition $U \subseteq \bigcap_{j \geq 1} Y + 2^{-j}U$. Since $2^{-j}U$ constitute a local basis it follows that $U \subseteq \overline{Y}$. Since Y is closed by (27) it follows that $U \subseteq Y$. Therefore Y contains $\bigcup_{j \geq 1} jU = X$, hence $Y = X$.)

$$\boxed{\text{locally compact} \leftrightarrow \text{finite dimensional}}.$$

3.2 Some famous examples of Frechet spaces

Here are some famous example of Frechet spaces appearing in different branches of mathematical analysis. None of these examples are normable, which gives a motivation to study TVS as a generalization of normed vector spaces.

1. $X := C(U)$, $U \subseteq \mathbb{R}^n$ nonempty open, the vector space of all continuous functions $f : U \to \mathbb{C}$. It is topologized via the separating family of seminorms

$$p_j(f) = \sup\{|f(x)| : x \in K_j\}, \quad j \in \mathbb{N},$$

   where K_j is an exhaustion of U by compacts [Lee, A.60], namely each K_j is compact and contained in the interior of K_{j+1} and $U = \bigcup K_j$. (Different exhaustions induce the same topology) This is called the **topology of uniform convergence on compacts**. (Also equal to the "compact-open topology" [Mun, 46.8].) We will show that:

$$\boxed{C(U) \text{ is Frechet, not locally bounded, not normable.}}$$

   We check these properties step by step:

- A local subbasis is given by $U_{jk} = \{p_j(f) < 1/k\}$, but since $p_1 \leq p_2 \leq \cdots$, a local basis is given by $U_j = \{p_j(f) < 1/j\}$.
- *Frechetness.* Since the topology is given by a countable separating family of seminorms it follows that X is locally convex and metrizable. A compatible invariant metric is $d(f,g) = \max \frac{2^{-j}p_j(f-g)}{1+p_j(f-g)}$. For completeness assume a Cauchy sequence f_α in X. This means $p_j(f_\alpha - f_\beta) \to 0$ as $\alpha, \beta \to \infty$ for every j, so appears the limit function $f : U \to \mathbb{C}$ such that $f_\alpha \to f$ uniformly on compacts. f is continuous because continuity can be checked locally and on each K_j the function f is a uniform limit of continuous functions.
- *Not locally bounded.* We show that no U_j is bounded. $A \subseteq X$ being bounded means that all p_k are bounded on A, namely there exist $M_k > 0$ such that $|f| < M_k$ on K_k for every $f \in A$. However every U_j contains (even C^∞) functions with arbitrary large p_{j+1} [Lee, 2.25].
- *Not normable.* Because X is not locally bounded.

Exercise: Show that $C(\mathbb{R})$ does not have the Heine-Borel property. (*Hint.* Consider the subset $\{f \in C(\mathbb{R}) : -1 \leq f \leq 1\}$, and think about the sequence $f_j = \sin(jx)$.)

2. $X := \mathrm{Holo}(D)$, $D \subseteq \mathbb{C}^m$ nonempty open, the vector space of all holomorphic (also called complex analytic) functions $D \to \mathbb{C}$. We accept the subspace topology $X \subseteq C(D)$, where D is pretended to be an open subset of $\mathbb{R}^{2m}$ in a natural way. We will show that:

$$\boxed{\text{Holo}(D) \text{ is Frechet, has Heine-Borel property, not locally bounded, not normable.}}$$

We check these properties step by step:

- *Completeness.* By Weierstrass theorem [Ahl, page 176][Hör-SCV, 1.2.5, 2.2.4] X is a closed subspace of $C(D)$, so complete.
- *Heine-Borel property.* Let $A \subseteq X$ be a closed and bounded subset. A being bounded means that there exists $M_j > 0$ such that $|f| < M_j$ on K_j for every $f \in A$ and every $j \in \mathbb{N}$. (Here K_j is an exhaustion of D by compacts.) By Montel's compactness theorem [Ahl, pages 224-5][Hör-SCV, 1.2.6, 2.2.5] (which is the adaption of Arzela-Ascoli theorem to the holomorphic setting) every sequence in A has a subsequence which converges uniformly on compacts; that limit function is holomorphic by Weierstrass theorem. Since A is closed it contains the limit function. This argument shows that A is sequentially compact, hence compact because X is metrizable.
- *Not locally bounded.* If X was locally bounded, because it has Heine-Borel property, it should have been locally compact, so of finite vector space dimension. However the monomials $z_1^{\alpha_1} \cdots z_m^{\alpha_m}$, $(\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m$, are in X and linearly independent. (Here $z_1, \ldots, z_m$ are standard coordinate functions of $\mathbb{C}^m$.)
- *Not normable.* Because X is not locally bounded.

3. $X := \mathcal{E}(U)$, $U \subseteq \mathbb{R}^n$ nonempty open, the vector space of all C^∞ functions $U \to \mathbb{C}$. It is topologized via the separating family of seminorms

$$p_j(f) = \sup\{|\partial^\alpha f(x)| : x \in K_j, |\alpha| \leq j\}, \quad j \in \mathbb{N},$$

where K_j is an exhaustion of U by compacts, and we are using the multi-index notations (page 3). We will show that:

$\boxed{\mathcal{E}(U) \text{ is Frechet, has Heine-Borel property, not locally bounded, not normable.}}$

We check these properties step by step:

- A local basis is given by $U_j = \{p_j(f) < 1/j\}$.
- *Frechetness.* Since the topology is given by a countable separating family of seminorms it follows that X is locally convex and metrizable. A compatible invariant metric is $d(f,g) = \max \frac{2^{-j}p_j(f-g)}{1+p_j(f-g)}$. For completeness assume a Cauchy sequence f_a in X. This means that $p_j(f_a - f_b) \to 0$ as $a, b \to \infty$ for each j, so each $\partial^\alpha f_a$ converges uniformly on compacts to some function g_α. Since $f_a \to g_0$ pointwisely (here 0 means multi-index $(0, \ldots, 0)$), it follows by a simple application of the mean value theorem for differentiation [Apo-A, 9.13] that each g_α is smooth and equals $\partial^\alpha g_0$, and that $f_a \to g_0$ in X.
- *Heine-Borel property.* Let $A \subseteq X$ be a closed and bounded subset. A being bounded means that there exists $M_j > 0$ such that $|\partial^\alpha f| < M_j$ on K_j for every $f \in A$, every $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq j$ and every $j \in \mathbb{N}$. For every j the inequality $|\partial^\alpha f| < M_j$ valid on K_j for $|\alpha| \leq j$, along with a simple application of the mean value theorem for differentiation, shows that $\{\partial^\beta f : f \in A\}$ is equicontinuous on K_{j-1} for every $\beta \in \mathbb{N}^n$ with $|\beta| \leq j-1$; therefore by Arzela-Ascoli theorem [Fol, 4.43] every sequence in A has a subsequence which converges with respect to p_j. By a Cantor diagonal argument one can deduce that every sequence in A has a subsequence which converges with respect to every p_j, namely in the topology of X. Since A is closed it contains that limit function. This argument shows that A is sequentially compact, hence compact because X is metrizable.
- *Not locally bounded.* If X was locally bounded, because it has Heine-Borel property, it should have been locally compact, so of finite vector space dimension. However the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $(\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, are in X and linearly independent. (Here $x_1, \ldots, x_n$ are standard coordinate functions of $\mathbb{C}^n$.)
- *Not normable.* Because it is not locally bounded.

4. $X := \mathcal{E}_K(U)$, $U \subseteq \mathbb{R}^n$ open subset, $K \subseteq U$ compact, the vector space of all C^∞ functions whose support $\overline{\{x \in U : f(x) \neq 0\}}$ is contained in K. We accept the subspace topology $X \subseteq \mathcal{E}(D)$. Exactly the same as in the case $\mathcal{E}(U)$ one can show that:

$\boxed{\mathcal{E}_K(U),\ \text{int}K \neq \emptyset, \text{ is Frechet, has Heine-Borel property, not locally bounded, not normable.}}$

That the vector space dimension of $\mathcal{E}_K(U)$ is infinite follows from the existence of smooth bump function [Lee, 2.25]. We assumed that the interior of K is nonempty just because otherwise $\mathcal{E}_K(U) = \{0\}$.

5. $\mathcal{S}$, the **Schwartz space**, the vector space of all C^∞ functions $f : \mathbb{R}^n \to \mathbb{C}$ which, together with all their derivatives, vanish at infinity faster that any power of $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, more precisely, $\sup_{x\in\mathbb{R}^n} |x|^j |\partial^\alpha f(x)| < \infty$ for every $j \in \mathbb{N}$ and $\alpha \in \mathbb{N}^n$. Some authors call them **rapidly decreasing functions** functions. This space is used extensively in Fourier analysis and distribution theory. It is topologized via the separating family of seminorms

$$p_j(f) = \sup\left\{(1+|x|^2)^j |\partial^\alpha f(x)| : x \in \mathbb{R}^n, \alpha \in \mathbb{N}^n, |\alpha| \le j\right\}, \quad j \in \mathbb{N}.$$

(Replacing $(1+|x|^2)^j$ with $|x|^j$ leads to the same topology.)

$\boxed{\mathcal{S} \text{ is Frechet, has Heine-Borel property, not locally bounded, not normable.}}$

The proof is exactly the same as the one for $\mathcal{E}(U)$. The vector space dimension of $\mathcal{S}$ is infinite because $x^\alpha \exp(-|x|^2)$, $\alpha \in \mathbb{N}^n$, are linearly independent members of $\mathcal{S}$.

Exercise: Let X be a TVS whose topology is given by a countable separating family of seminorms p_j, $j \in \mathbb{N}$. Also assume that $p_1 \le p_2 \le \ldots$. (Note that this assumption holds for all examples of this section.) Give a direct argument that if X is normable then all p_j for sufficiently large j are equivalent. Use this proposition to prove that the Schwarz space $\mathcal{S}$ is not normable. (*Hint.* Review the proof of 16.)

3.3 An F-space which supports no continuous linear functional

Let $X := L^p([0,1])$, $p \in (0,1)$, be the space of functions $[0,1] \to \mathbb{C}$ which are L^p-integrable with respect to the Lebesgue measure. We topologize it via the metric $d(f,g) = \int |f-g|^p$. (Triangle inequality follow from $a^p + b^p \ge (a+b)^p$ valid for $a, b \ge 0$.)

$\boxed{\begin{array}{l} L^p([0,1]),\ p \in (0,1), \text{ is } F\text{-space, locally bounded, not locally convex } (X \text{ and } \emptyset \text{ are} \\ \text{the only convex opens), not normable, with no nonzero continuous linear functional.} \end{array}}$

We check these properties step by step:

- *Completeness.* Adapt the proof of the completeness of L^p, $p \in [1, \infty)$.
- *Locally bounded.* $U_r := \{f \in X : d(f,0) < r\}$, $r > 0$, is a local basis, and $U_1 = r^{-1/p} U_r$. It follows that U_1 is bounded.
- *Not normable.* Because it is not locally convex.

- *X and $\emptyset$ are the only convex opens.* Assuming a nonempty convex open $V \subseteq X$ we show that $V = X$. We can assume that V contains the origin. Since V is open it follows that $V \supseteq U_r$ for some $r > 0$. Fix some $f \in X$. Since $0 < p < 1$ one can find $n \in \mathbb{N}$ large enough such that $n^{p-1}d(f,0) < r$. By continuity one can inductively find $x_0 = 0 < x_1 < \ldots < x_{n-1} < x_n = 1$ such that $\int_{x_{j-1}}^{x_j} |f|^p = d(f,0)/n$ for every $j = 1, \ldots, n$. Set $g_j := nf1_{(x_{j-1},x_j]}$. Then

$$d(g_j, 0) = \int_{x_{j-1}}^{x_j} |f|^p n^p = n^{p-1}d(f,0) < r,$$

so $g_j \in U_r \subseteq V$. On the other hand $f = (g_1 + \cdots + g_n)/n$. Since g_j belongs to convex V it follows that $f \in V$.

- Let α be a continuous linear functional. $B_\epsilon := \{a \in \mathbb{F} : |a| < 1\}$ is open and convex for every $\epsilon > 0$, so is $\alpha^{-1}(B_\epsilon) \subseteq X$, therefore $\alpha^{-1}(B_\epsilon) = X$. This means that $\alpha \equiv 0$. is open and convex, hence equals X be the previous part.

Remark 24. Recall the fundamentals theorems of Banach spaces: Uniform boundedness principle, Open mapping theorem, Inverse mapping theorem and closed graph theorem (Theorem 5). They have analogues for TVSs. Statements of the last three are exactly as before unless "Banach space" is replaced with "F-space" [Rud-FA, 2.12, 2.15]. Here is the statement for the uniform boundedness principle [Rud-FA, 2.4, 2.6]: *A family $T_\alpha : X \to Y$, $\alpha \in A$, of bounded operators from F-space X to TVS Y is uniformly equibounded (namely for every bounded subset $B \subseteq X$ there exists a bounded subset $B' \subseteq Y$ such that $T_\alpha(B) \subseteq B'$ for every $\alpha \in A$) if and only if it is pointwisely equibounded (namely $\{T_\alpha(x) : \alpha \in A\}$ is bounded for every $x \in X$).* Another good reference is [DS, chapter 2]. ■

Chapter 4

Duality theory I: Hahn-Banach theorem

References: [Rud-FA, chapter 3][Bre, chapter 1].

Theorem 25 (Hahn-Banach)**.** *The followings are true:*

1- Controlled extension from linear subspaces of real vector spaces. Let X be a real vector space, Y a linear subspace, $p : X \to \mathbb{R}$ a **sublinear functional** *(namely, $p(x+y) \leq p(x)+p(y)$ and $p(ax) = ap(x)$ for every $x, y \in X$ and $a \geq 0$; These are called subadditivity and positive homogeneity. For example every seminorm is sublinear.), and f a linear functional on Y which is dominated by p (namely, $f(x) \leq p(x)$ for every $x \in Y$.). Then f can be extended to a linear functional F on X which is still dominated by p.*

2- Controlled extension from linear subspaces of real or complex vector spaces. Let X be a vector space, Y a linear subspace, $p : X \to [0, \infty)$ a seminorm, and f a linear functional on Y with is dominated by p (namely, $|f(x)| \leq p(x)$ for every $x \in Y$). Then f can be extended to a linear functional F on X which is still dominated by p.

2′- Continuous extension from linear subspaces of normed vector spaces. Every continuous linear functional on a linear subspace of a normed vector space can be extended to a continuous linear functional on the whole space with the same norm.

3- Separation theorem for normed vector spaces. Let X be a normed vector space, Y a closed linear subspace and $x \in X \setminus Y$. Then there exists a continuous linear functional F on X such that $F|_Y \equiv 0$, $F(x) = \inf_{y \in Y} \|x - y\| > 0$ and $\|F\| = 1$. In geometric terms, in a normed vector space X, a point x and a closed linear subspace Y that are disjoint can be strictly separated by closed hyperplanes in the sense that there exists a continuous liner functional F on X and $K \in \mathbb{R}$ such that $\mathrm{Re}F(y) < K < \mathrm{Re}F(x)$ for every $y \in Y$. Specially, X^* separate points on X in the sense that for every two distinct points $x, y \in X$ there exits $F \in X^*$ such that $F(x) \neq F(y)$.

3′- Closure theorem. Let X be a normed vector space, Y a linear subspace and $x \in X$. Then $x \in \overline{Y}$ if and only if every continuous linear functional on X which vanishes on Y also vanishes on x. In other words, $\overline{Y} = {}^{\perp}Y^{\perp}$ where the **annihilators** *of subsets $A \subseteq X$ and $B \subseteq X^*$ are defined by $A^{\perp} = \{\alpha \in X^* : \alpha(x) = 0, \forall x \in A\}$ and ${}^{\perp}B = \{x \in$*

$X : \alpha(x) = 0, \forall \alpha \in B\}$. Specially, Y is dense in X if and only if there is no nonzero continuous functional on X which vanishes on Y (in notations: $Y^\perp = \{0\}$.)

4- Separation theorem for LCTVSs; generalization of (3). Let A and B be two disjoint nonempty convex subsets of a TVS X. If A is open then A and B **can be separated by closed hyperplanes** *in the sense that there exists $F \in X^*$ and $K \in \mathbb{R}$ such that $\mathrm{Re}F(a) < K \leq \mathrm{Re}F(b)$ for every $a \in A$ and $b \in B$. If A is compact, B is closed and X is locally convex then A and B* **can be strictly separated by closed hyperplanes** *in the sense that there exists $F \in X^*$ and $K_1, K_2 \in \mathbb{R}$ such that $\mathrm{Re}F(a) < K_1 < K_2 < \mathrm{Re}F(b)$ for every $a \in A$ and $b \in B$. Specially, if X is a LCTVS then X^* separate points on X in the sense that for every two distinct points $x, y \in X$ there exits $F \in X^*$ such that $F(x) \neq F(y)$.*

4′- Closure theorem; generalization of (3′). The analogues of (3′) holds if "normed vector space" is replaced by "LCTVS".

5- Continuous extension from linear subspaces of LCTVSs. Every continuous linear functional on a linear subspace of a LCTVS can be extended to a continuous linear functional on the whole space.

A linear subspace Y of a normed vector space X is dense if and only there is no nozero continuous linear functional on X which vanishes on Y.

Two disjoint nonempty closed convex subsets of a LCTVS, at least one of them compact, can be strictly separated by closed hyperplanes.

Proof. (1) Consider the set of all pairs (Y_1, f_1) where f_1 is a linear extensions of f to a linear subspace $Y_1 \subseteq X$ containing Y which is dominated by p, and partially order it by declaring $(Y_1, f_1) \leq (Y_2, f_2)$ if and only if $Y_1 \subseteq Y_2$ and $f_1 = f_2|_{Y_1}$. In this poset each chain (Y_α, f_α) has an upper bound: (Z, g) where $Z = \bigcup Y_\alpha$ and $g|_{Y_\alpha} = f_\alpha$. By Zorn lemma there is a maximal element (Z, F). We will show that $Z = X$. Assuming some $x \in X \setminus Z$ we need to refute maximality of (Z, F), namely find a linear extension of F to $Z + \mathbb{R}x$ which is dominated by p. This is equivalent to finding some $b \in \mathbb{R}$ such that setting $F(x) = b$ then we have $F(z) + \lambda b = F(z + \lambda x) \leq p(z + \lambda x)$ for every $z \in Z$ and $\lambda \in \mathbb{R}$. This happens exactly when

$$\frac{p(z_1 - \lambda_1 x) - F(z_1)}{-\lambda_1} \leq \frac{p(z_2 + \lambda_2 x) - F(z_2)}{\lambda_2}, \quad \forall z_1, z_2 \in Z, \forall \lambda_1, \lambda_2 \in \mathbb{R}_+,$$

which can be rewritten as

$$\lambda_2 F(z_1) + \lambda_1 F(z_2) \leq p(\lambda_1 z_2 + \lambda_1\lambda_2 x) + p(\lambda_2 z_1 - \lambda_1\lambda_2 x), \quad \forall z_1, z_2 \in Z, \forall \lambda_1, \lambda_2 \in \mathbb{R}_+.$$

This holds by the sublinearity of p and that it dominates F on Z.

(2) Assuming the real case, note that $f(x) \leq p(x)$ for every x is equivalent to $|f(x)| \leq p(x)$ for every x, by replacing x with $-x$. Therefore (1) can be applied. Now the complex case. Let $f = g + \sqrt{-1}h$ be the decomposition of f into real-valued functionals g, h. Then g, h are $\mathbb{R}$-linear, and $f(\sqrt{-1}x) = \sqrt{-1}f(x)$ is equivalent to $h(x) = -g(\sqrt{-1}x)$. Since $g(x) \leq |f(x)| \leq p(x)$ on Y it follows by (1) that g can be extended to an $\mathbb{R}$-linear functional G on X such that $G(x) \leq p(x)$ on X. Clearly, $F(x) = G(x) - \sqrt{-1}G(\sqrt{-1}x)$

is a $\mathbb{C}$-linear functional on X which extends f. It remains to check that $|F(x)| \leq p(x)$. Fixing x, and replacing x with $\exp(\sqrt{-1}\theta)x$, $\theta \in \mathbb{R}$, in $G(x) \leq p(x)$ we have $G(x)\cos\theta + G(\sqrt{-1}x)\sin\theta \leq p(x)$. Since this is true for every θ it follows that $|F(x)|^2 = G(x)^2 + G(\sqrt{-1}x)^2 \leq p(x)^2$.

(2′) Immediate from (2) by taking p to be the norm of X multiplied by $\|f\|$.

(3) Since Y is closed and does not contain x it follows that the distance $\delta := \inf_{y\in Y} \|x - y\|$ between x and Y is strictly positive. The linear functional $f : Y + \mathbb{F}x \to \mathbb{F}$, $y + ax \mapsto a\delta$, has norm equal to one:

$$|f(y + ax)| = |a|\delta \leq |a|\|y/a + x\| \leq \|y + ax\|, \quad \forall y \in Y.$$

By (2′), f can be extended to a continuous linear functional F on X with the same norm, namely 1. F has the other two desired properties.

(3′) Only if part is trivial. If $x \in X \setminus \overline{Y}$ then by (3) there exists a continuous linear functional F on X which vanishes on $\overline{Y}$ (hence on Y) but not at x.

(4) We assume $\mathbb{F} = \mathbb{R}$, because the complex case is then deduced having in mind the correspondence between real and complex functionals that already appeared in the proof of (2): $f(x) = \mathrm{Re}f(x) - \sqrt{-1}\mathrm{Re}f(\sqrt{-1}x)$. For the first statement assume A open. Fix $a_0 \in A$, $b_0 \in B$, and set $x_0 := b_0 - a_0$, $C := A - B + x_0$. Then C is a convex neighborhood of the origin in X. Note that:

- C is absorbing because it is a neighborhood of the origin (Section 3.1.(9)).
- Since C is convex and absorbing it follows that for every $x \in X$ the set $\{t > 0 : x \in tC\}$ has the property that if it contains t then it contains $[t, \infty)$.
- Just because C is convex and absorbing one can easily check that its gauge functional $p = \mu_C : X \to [0, \infty)$, $x \mapsto \inf\{t > 0 : x \in tC\}$, is sublinear [Rud-FA, 1.35].
- Since C is also open we have $C = \{x \in X : p(x) < 1\}$ (Section 3.1.(14)).
- $A \cap B = \emptyset$ implies that $x_0 \notin C$, so $p(x_0) \geq 1$.

Consider the linear functional $f : \mathbb{R}x_0 \to \mathbb{R}$, $tx_0 \mapsto t$. f is dominated by p because if $t > 0$ then $f(tx_0) = t \leq tp(x_0) = p(tx_0)$, and if $t < 0$ then $f(tx_0) = t < 0 \leq p(tx_0)$. By (2), f can be extended to a linear functional F on X dominated by p. Specially, $F \leq p < 1$ on C, hence by linearity $F > -1$ on $-C$, so $-1 < F < 1$ on the neighborhood $C \cap -C$ of 0. The continuity of F follows. For any $a \in A$ and $b \in B$ we have

$$F(a) - F(b) + 1 = F(a - b + x_0) \leq p(a - b + x_0) < 1,$$

because $a - b + x_0 \in C$. This gives $F(a) < F(b)$. It follows that $F(A)$ and $F(B)$ are disjoint convex subsets of $\mathbb{R}$, with $F(A)$ to the left of $F(B)$. On the other hand $F(A)$ is open because every nonzero continuous linear functional on a TVS is clearly an open map. Therefore $K := \sup F(A)$ works.

For the second statement, first using Section 3.1.(6) find neighborhood U of 0 such that $(A + U) \cap B = \emptyset$. Since X is LCTVS one can assume that U is convex. Applying the previous part to separate $A+U$ and B, one finds $F \in X^*$ such that $F(A+U)$ and $F(B)$ are disjoint convex subsets of $\mathbb{R}$ with $F(A+U)$ open and to the left of $F(B)$. We are done because $F(A)$ is a compact subset of $F(A+U)$.

(4′) Only if part is trivial. If $x \in X \setminus \overline{Y}$ then by (4) there exists a continuous linear functional F on X such that $\mathrm{Re}F(x) < K < \mathrm{Re}F(z)$ for every $z \in \overline{Y}$. Since $\overline{Y}$ is closed under scalar multiplication it follows that $F|_{\overline{Y}} \equiv 0$ (hence $F|_Y \equiv 0$); however $F(x) \neq 0$.

(5) Assume continuous linear functional f on linear subspace Y of LCTVS X. Putting the trivial case $f \equiv 0$ aside, one can find $y \in Y$ with $f(y) = 1$. By continuity, y does not belong to the closure of $B := \mathrm{Ker}_f$ in X, so by (4) there exists $F \in X^*$ and $K \in \mathbb{R}$ such that $\mathrm{Re}F(y) < K < \mathrm{Re}F(b)$ for every $b \in B$. Since $F(B)$ is a linear subspace of $\mathbb{F}$, it implies that and $F(y) \neq 0$ and $F|_B \equiv 0$. such that $F(y) = 1$ and $F|_K \equiv 0$. Dividing F by $F(y)$, we may assume that $F(y) = 1$. Since F and f agree on the codimension-1 linear subspace B of Y as well as on a single point y not in B, it follows that $F = f$ on whole Y: For every $z \in Y$, since $z - f(z)y \in B$, it follows that $F(z) = F((z - f(z)y) + f(z)y) = 0 + f(z)F(y) = f(z)$. ■

Remark 26. Regarding Theorem 25.(4), if X has finite dimension then there are many elementary proofs in the literature [Web, 2.4.6, 2.4.10][Hör-C, 2.1.11][Rock, 11.4].

Exercise: If $x_1, \dots, x_n$ are finitely many linear independent vectors in a normed vector space X and $a_1, \dots, a_n$ are arbitrary scalars then there exists a continuous linear functional F on X such that $F(x_j) = a_j$ for every j.

Here are some corollaries:

Theorem 27. *Let X, Y be normed vector spaces.*

(1) Continuous linear functionals can be used to compute the norm of elements $x \in X$ in the sense that

$$\|x\| = \sup\{|\langle x, \alpha\rangle| : \alpha \in X^*, \|\alpha\| \leq 1\}.$$

(2) Continuous linear functionals can be used to compute the norm of bounded operators $T \in B(X,Y)$ in the sense that

$$\|T\| = \sup\{|\langle Tx, \beta\rangle| : x \in X, \|x\| \leq 1, \beta \in Y^*, \|\beta\| \leq 1\}.$$

(3) $\|T\| = \|T^*\|$ for every bounded operator $T : X \to Y$.

(4) The mapping $X \to X^{**}$, $x \mapsto \widehat{x}$, given by $\widehat{x}(\alpha) = \alpha(x)$ is an isometric isomorphism onto its range. This map is usually denoted by J and called the **natural embedding** *of X into X^{**}. (Recall that the closure of the range is the* **completion** *of X (page 8). Note that X^{**} is always complete so is every closed subspace of it.). X is called* **reflexive** *if this map is surjective.*

Proof. (1) $\geq$ is because $|\alpha(x)| \leq \|\alpha\|\|x\| \leq \|x\|$. $\leq$ is because by Theorem 25.(3), when $Y = 0$ and $x \neq 0$, there exists $\alpha \in X^*$ such that $\alpha(x) = \|x\|$ and $\|\alpha\| = 1$.

(2) Immediate by Applying (1) to $\|Tx\|$ in $\|T\| = \sup_{\|x\| \leq 1} \|Tx\|$

(3) Immediate by Applying (2) to $\|T^*\| = \sup_{\|\alpha\| \leq 1} \|T^*\alpha\| = \sup_{\|\alpha\| \leq 1, \|x\| \leq 1} |\langle x, T^*\alpha\rangle|$.

(4) The map is clearly linear. That it is an isometry is exactly (1). Every surjective isometry map clearly have continuous inverse. ■

Remark 28. There are examples of non-reflexive Banach spaces X that are isometrically isomorphic to X^{**} [Jam].

Example 29. (1) $L^p(X,\mu)$ is reflexive if $1 < p < \infty$ [Fol, 6.16]. L^1 (also L^∞) is not reflexive unless it is finite-dimensional [Bre, chapter 5]. (2) We have canonical isometric isomorphisms $c_0^* \cong l^1$ and $(l^1)^* \cong l^\infty$, both given by coupling $(f,g) \mapsto \sum_{j=0}^{\infty} f(j)g(j)$. Both are straightforward to check directly [Dou, page 7], but they can be deduced from big duality theorems [Fol, 7.17 and page 225] and [Fol, 6.15]. Specially, c_0 is not reflexive. See also Example 42. ■

Application 30 (Runge's approximation theorem)**.** *For every open $D \subseteq \mathbb{C}$ and compact $K \subseteq D$ the followings are equivalent:*

(1; topological condition) K adds no hole to D, in the sense that $D \setminus K$ has no component compactly supported in D.

(2; functional analysis condition) $\mathcal{O}(D)$ is dense in $\mathcal{O}(K)$, in the sense that every holomorphic function on K can be uniformly approximated on K by holomorphic functions on D.

(3; function theory condition) K is holomorphically convex in D, in the sense that for any $z \in D \setminus K$ there exists some holomorphic function f on D such that $|f(z)| > \sup_K |f|$.

Proof. (2 or 3⇒1) Assume (1) fails. Then $D \setminus K$ has a component O which is compactly supported in D. Note that $\partial O \subseteq K$. By the maximum principle

$$\|f\|_{\overline{O}} \leq \|f\|_K, \quad \forall f \in \mathcal{O}(D), \tag{4.1}$$

which contradicts (3) for any $z \in O$. Now let (2) hold. Fix $\zeta \in O$. Applying (2) to $f(z) := (z-\zeta)^{-1} \in \mathcal{O}(K)$ gives a sequence f_n of holomorphic functions on D which converge uniformly on K to f. Applying (4.1) to $f_n - f_m$ shows that f_n converges uniformly on $\overline{O}$ to some limit function F. Note that F is holomorphic on O, continuous on $\overline{O}$, and equals f on ∂O namely $(z-\zeta)F(z) = 1$ on ∂O. This latter identity persists on $\overline{O}$ by the maximum principle applied to $z \mapsto (z-\zeta)F(z) - 1$. This gives a contradiction when $z = \zeta$.

(1⇒2) Fix an arbitrary $f \in \mathcal{O}(K)$. Consider f as an element of the space $C(K)$ of continuous functions on K equipped with uniform norm. Since the dual of $C(K)$ is given by (regular Borel) measures, according to the Hahn-Banach theorem, we need to check that any measure μ on K which is orthogonal to $\mathcal{O}(D)$ (namely $\int g d\mu = 0$ for all $g \in \mathcal{O}(D)$) is also orthogonal to f. Let ψ be a smooth bump function compactly supported on some neighborhood of K where f is holomorphic on, and ψ equals 1 on some neighborhood of K. By the Cauchy-Pompeiu formula

$$f(z) = \frac{1}{2\pi\sqrt{-1}} \int \frac{f(\zeta)\psi_{\overline{\zeta}}(\zeta)}{\zeta - z} d\zeta \wedge d\overline{\zeta}, \quad \forall z \in K,$$

where $d\zeta \wedge d\overline{\zeta}$ is the Lebesgue measure on $\mathbb{C}$, multiplied by $-2\sqrt{-1}$. (Alternatively, one can find a cycle γ in $D \setminus K$ such that the Cauchy formula $f(z) = (2\pi\sqrt{-1})^{-1} \int_\gamma f(\zeta)(\zeta - z)^{-1} d\zeta$ holds for every $z \in K$.) Therefore, by the Fubini's theorem:

$$\int f(z) d\mu(z) = \frac{1}{2\pi\sqrt{-1}} \int f(\zeta)\psi_{\overline{\zeta}}(\zeta)\varphi(\zeta) d\zeta \wedge d\overline{\zeta},$$

where $\varphi(\zeta) = \int (\zeta - z)^{-1} d\mu(z)$. It suffices to show that the function φ defined on $\mathbb{C} \setminus K$ is identically zero. Fix an arbitrary point $z \in \mathbb{C} \setminus K$. Clearly φ is holomorphic. It also vanishes on the unbounded component of $\mathbb{C} \setminus K$ because $(\zeta - z)^{-1}$ is a uniform sum of monomials $z^n \in \mathcal{O}(D)$ on $|\zeta| \geq 2 \sup_{w \in K} |w|$. Let O be an arbitrary bounded component of $\mathbb{C} \setminus K$. Because of our topological assumption O intersects $\mathbb{C} \setminus D$, so let ζ_0 be a point in the intersection. Then $\partial^k \varphi / \partial \zeta^k (\zeta_0) = (-1)^k k! \int (\zeta_0 - z)^{-k-1} d\mu(z)$ vanishes because $(\zeta_0 - z)^{-k-1}$ is holomorphic on D. By the identity theorem φ vanishes on whole O.

(1 and 2$\Rightarrow$3) Fix $z \in D \setminus K$. Choose a closed disc L centered at z with $L \subseteq D \setminus K$. The components of $D \setminus (K \cup L)$ are the same as those of $D \setminus K$ apart from the fact that L has been removed from exactly one of them. Therefore $K \cup L$ adds no hole to D. Applying (2) to the function which is 0 in a neighborhood of K and is 1 in a neighborhood of L gives $f \in \mathcal{O}(D)$ such that $\|f\|_K < 2^{-1}$ and $\|f - 1\|_L < 2^{-1}$. This f satisfies (3). ∎

Remark 31. (1) Theorem 30 is the version of Runge's approximation theorem which appeared in [Hör-SCV]. The more famous version says: *Given compact $K \subseteq \mathbb{C}$ and any $P \subseteq \mathbb{C}$ which contains at least one point in each bounded component of $\mathbb{C} \setminus K$, every holomorphic function on K can be uniformly approximated on K by rational functions with poles in P.* This latter version can be proved with exactly the same techniques [Rud-RCA, 13.6]; An elementary proof is given in [Sar, page 115][Gam, page 344]. (2) The power series representation of holomorphic functions shows that holomorphic functions on $\mathbb{C}$ can be uniformly approximated on compacts by polynomials. Therefore, for the special case $D = \mathbb{C}$, one deduces from Theorem 30 that: *For $K \subseteq \mathbb{C}$ compact, $\mathbb{C} \setminus K$ is connected if and only if every holomorphic function on K can be uniformly approximated on K by polynomials.* This is another useful version of Runge's theorem [Rud-RCA, 13.7, 13.8]. ∎

Here is an application of Hahn-Banach theorem to PDEs. Usually the existence of Green function for the Laplacian is proved via the solvability of Dirichlet problem; however Lax [Lax-G] found a detour using Hahn-Banach extension theorem:

Application 32 (Existence of Greens function for smooth domains)**.** *Let $U \subseteq \mathbb{R}^2$ be a bounded open with C^2 boundary ∂U. Then: (1) For every (field point) $y \in U$ there exists a function $g(x, y)$ which is harmonic with respect to $x \in U$ and continuous up to the boundary with boundary value $(2\pi)^{-1} \log |x - y|$.*

(2) The Dirichlet problem "For every $f \in C(\partial U)$ find $u \in C^2(\overline{U})$ such that $\Delta u = 0$ on U and $u = f$ on ∂U" can be solved via $u(y) = \int_{\partial U} f(x) G_n(x, y) dx$, where G_n is the derivative of $G(x, y) := g(x, y) - (2\pi)^{-1} \log |x - y|$ in the outward normal direction.

Proof. (1) Consider the space $C(\partial U)$ of continuous functions on ∂U equipped with the uniform norm, let $A(\partial U)$ be the subspace consisting of the boundary values of harmonic functions in D which are continuous up to the boundary. By maximum principle the linear functional

$$\alpha_y : A(\partial U) \to \mathbb{R}, \quad f \mapsto f(y),$$

is bounded by 1, so can be extended to a linear functional on $C(\partial U)$, denoted again by α_y. For every $\xi \in \mathbb{R}^2 \setminus \partial U$consider the function k_ξ defined on ∂U by

$$k_\xi(z) = k(z, \xi) = (2\pi)^{-1} \log |z - \xi|, \quad z \in \partial U.$$

We will prove that

$$g(\xi, y) := \alpha_y(k_\xi), \quad \xi \in \mathbb{R}^2 \setminus \partial U,$$

when restricted to $\xi \in U$, works as our Green function. That $g(\xi, y)$ is harmonic with respect to ξ is immediate from the two facts that k_ξ is harmonic with respect to ξ and α_y is a continuous linear functional. On the other hand, for every $\xi \in \mathbb{R}^2 \setminus \overline{U}$, since $k_\xi \in A(\partial U)$ it follows that $g(\xi, y) = k_\xi(y) = (2\pi)^{-1} \log|\xi - y|$. Therefore, we are left to show that $g(\xi, y)$ depends continuously on ξ as ξ crosses the boundary ∂U. To show this let ξ be a point in U close to the boundary, and ξ' be its reflection across the boundary in the sense that $\xi + \xi' = 2z_0$, where z_0 is the point on ∂U of smallest distance to ξ. (This can be proved by the implicit function theorem using C^2 assumption [KP, section 4.4].) Consider

$$g(\xi, y) - g(\xi', y) = \frac{1}{2\pi} \alpha_y \left(\log \frac{|z - \xi|}{|z - \xi'|} \right).$$

Since ∂U is C^1 and compact it follows that $|z - \xi|/|z - \xi'| \to 0$, uniformly for all points $z \in \partial U$, as $\xi \to \partial U$. Since α_y is continuous it follows that $g(\xi, y) - g(\xi', y) \to 0$ as $\xi \to \partial U$.

(2) G satisfies $\Delta G = \delta(x - y)$ and $G|_{\partial U} \equiv 0$. Green reciprocity formula

$$\int_{\partial U} u G_n - G u_n = \int_U u \Delta G - G \Delta u,$$

valid for every $G, u \in C^2(\overline{U})$ reduces to $u = \int_{\partial U} u G_n$. ∎

Here is a simple application of Hahn-Banach separation that will be used later.

Theorem 33 (A variation of Hahn-Banach separation). *If B is a convex balanced closed subset of a LCTVS X and $x \in X \setminus B$ then there exists $F \in X^*$ such that $F(x) > 1$ but $|F| \leq 1$ on B.*

Proof. By Theorem 25 there exists $G \in X^*$ such that $\mathrm{Re} G(x) > K > \mathrm{Re} G(y)$ for some $K \in \mathbb{R}$ and every $y \in B$. Let $G(x) = r \exp(\sqrt{-1}\theta)$, $r > 0$, $\theta \in \mathbb{R}$. Since B is balanced it follows that $\overline{G(B)}$ is balanced, so it is a closed disk of radius $s \in (0, r)$ around the origin in $\mathbb{F}$. Clearly, $F := s^{-1} \exp(-\sqrt{-1}\theta) G$ works. ∎

Chapter 5

Duality theory II: Weak and weak star topologies

References: [Rud-FA, chapter 3][DS, chapter 5][Roy, chapters 14-5][Die].

In the study of spaces of functions there are topologies– different from the original topology– that are naturally induced by different classes of continuous linear functionals; these are weak and weak-star topologies to be discussed in this chapter. These topologies are rarely metrizable (Theorem 41) but are locally convex most of the time (Theorem 34), so Hahn-Banach theorem can be applied to them. We start with the definitions of these topologies:

1. Recall this from topology [Mun, section 13]: Let X be a set and $\mathcal{S}$ be a family of subsets of X. There is a smallest[1] topology on X such that all elements of $\mathcal{S}$ are open. (Smallest in the sense that opens of this topology are opens of all the other topologies such that all elements of $\mathcal{S}$ are open. Sometimes the term **weakest** is used instead of "smallest".) Opens of this topology are arbitrary unions of finite intersections of elements of $\mathcal{S}$. $\mathcal{S}$ is called a **subbasis** for this topology.

2. Let X be a set and $\mathcal{F}$ be a family of maps $f_\alpha : X \to Y_\alpha$, $\alpha \in A$, from X to topological spaces Y_α. The **weak topology on X generated by $\mathcal{F}$** is the smallest topology on X such that all f_α are continuous. In other words, it is the topology on X with subbasis consisting of all $f_\alpha^{-1}(U_\alpha)$, $\alpha \in A$, $U_\alpha \subseteq Y_\alpha$ open. Another description: A net $(x_i)_{i\in I}$ in X converges $x \in X$ in this topology if and only if $f_\alpha(x_i)$ converges $f_\alpha(x)$ in the topology of Y_α for every $\alpha \in A$. When all Y_α is Hausdorff one can easily show that the weak topology on X is Hausdorff if and only if $\mathcal{F}$ separate points in X in the sense that if x, y are two distinct points in X then there exist $\alpha \in A$ such that $f_\alpha(x) \neq f_\alpha(y)$.

3. Let X be a TVS, with dual space X^*, the space of continuous linear functionals on X. The **weak topology** on X is the weak topology generated by X^*. Another description:

[1] On a set X, topology $\mathcal{T}$ is called smaller than topology $\mathcal{T}'$ if every open of $\mathcal{T}$ is an open in $\mathcal{T}'$.

A net $(x_i)_{i\in I}$ in X converges $x \in X$ in this topology if and only if $\alpha(x_i) \to \alpha(x)$ for every $\alpha \in X^*$. Another description: A local basis at $x \in X$ for this topology is given by $\{y \in X : |\alpha_j(x-y)| < \epsilon, j = 1, \ldots, n\}$, $\alpha_j \in X^*$, $n \in \mathbb{N}$, $\epsilon > 0$.

Exercise: Show that a subset $A \subseteq X$ of a TVS is weakly bounded (namely it is bounded in the sense of Chapter 3.(8), when we put the weak topology on X) if and only if each $F \in X^*$ is bounded on A in the sense that there exists $C_F > 0$ such that $|F(x)| \leq K_F$ for every $x \in A$.

4. Let X be a TVS. The **weak star** (or **weak***, **weak**$*$) **topology** on X^* is the weak topology generated by $\{\widehat{x} : x \in X\}$ where $\widehat{x} : X^* \to \mathbb{F}$ denotes the linear functional acting by $\alpha \mapsto \alpha(x)$. Another description: A net $(\alpha_i)_{i\in I}$ in X^* converges $\alpha \in X^*$ in this topology if and only if $\alpha_i(x) \to \alpha(x)$ for every $x \in X$. Another description: A local basis at $\alpha \in X^*$ for this topology is given by $\{\beta \in X^* : |(\beta-\alpha)(x_j)| < \epsilon, j = 1, \ldots, n\}$, $x_j \in X$, $n \in \mathbb{N}$, $\epsilon > 0$.

When X is a normed vector space there are three topologies on X^*: weak star topology, weak topology and the normed topology (also called strong topology) in increasing order of strength. (Refer Proposition 35.)

Theorem 34 (When weak topologies are locally convex)**.** *(1) If X is a vector space and $\mathcal{F}$ is a vector space of linear functional on X which separate points of X then the weak topology on X generated by $\mathcal{F}$ makes X into a LCTVS whose dual space equals $\mathcal{F}$.*

(2) A LCTVS X equipped with the weak topology is a LCTVS whose dual space is X^*.

(3) If X is a TVS then the dual of X equipped with the weak-star topology is a LCTVS whose dual space equals X, where each $x \in X$ is seen as a linear functional on X^* acting by $\alpha \mapsto \alpha(x)$.

Weakly continuous linear functionals are known if the weak topology is given by a separating vector space.

Proof. (1) Topology is Hausdorff because $\mathcal{F}$ is separating. By linearity of elements of $\mathcal{F}$ translations are homeomorphisms. A local basis at 0 is given by

$$U = \{x \in X : |f_1(x)| < \epsilon, \ldots, |f_n(x)| < \epsilon\}, \quad n \in \mathbb{N}, f_1, \ldots, f_n \in \mathcal{F}, \epsilon > 0.$$

That vector space operations are continuous are straightforward to check. By linearity, each U is convex, so X is a LCTVS. It remains to check that $X^* = \mathcal{F}$. $\supseteq$ is trivial. For the other containment assume a linear functionals F on X which is continuous with respect to the weak topology generated by $\mathcal{F}$. By continuity, for every $\delta > 0$ there exists $f_1, \ldots, f_n \in \mathcal{F}$ and $\epsilon > 0$ such that $|F(x)| < \delta$ whenever $|f_j(x)| < \epsilon$ for all j. This implies $\bigcap \mathrm{Ker}_{f_j} \subseteq \mathrm{Ker}_F$, so by linear nullstellensatz (page 21) we have $F = \sum C_j f_j$, $C_j \in \mathbb{F}$. Then $F \in \mathcal{F}$ because $\mathcal{F}$ is a vector space.

(2) Immediate from (1). Note that since X is locally convex X^* separates points of X by Theorem 25.(4).

(3) In (1) replace X by X^* and set $\mathcal{F} := X$. Note that by the very definition, X separate points of X^*. ∎

Proposition 35. *Let X be a normed space. Then:*

(1) Weak and strong topologies on X coincide if and only if the vector space dimension of X is finite.

(2) Weak and weak-star topologies on X^* coincide if and only if X is reflexive.

Proof. (1) Let x_j, $j = 1, \dots, n$ be a basis for X. Consider $\alpha_j \in X^*$, $\sum a_j x_j \mapsto a_j$. Then $\{x \in X : |\alpha_j(x-y)| < \epsilon, \forall j = 1, \dots, n\}$, $\epsilon > 0$, $y \in X$, is a basis of opens for weak topology, and also a basis of opens for the topology induced by l^1 norm on X. Conversely, assume X is of infinite dimension. Then every nonempty basic weak neighborhood $\{|\alpha_j(x)| < \epsilon, j = 1, \dots, n\}$ of 0 is unbounded: one can inductively find infinitely many directions along which all α_j vanish. Therefore no such neighborhood is contained in the open unit ball of the strong topology.

(2) If part is trivial. For only if part, assume continuous linear functional F on X^*. Clearly, F is also continuous with respect to weak topology on X^*, hence continuous with respect to weak-star topology on X, hence of the form $\widehat{x}$, $x \in X$, by Theorem 34. ∎

A Banach space X is called **uniformly convex** if for every $\epsilon > 0$ there exists $\delta > 0$ such that for every $x, y \in X$ we have the following implication:

$$\|x\| \leq 1, \|y\| \leq 1, \|x-y\| > \epsilon \Rightarrow \|(x+y)/2\| < 1-\delta.$$

Equivalently, for every sequences x_j and y_j with $\|x_j\| = \|y_j\| = 1$ and $\|(x_j + y_j)/2\| \to 1$ we have $\|x_j - y_j\| \to 0$. Intuitively, if we slide a ruler of length ϵ in the unit ball of X then its midpoint must stay within a ball of radius $1-\delta$ for some $\delta > 0$. Figuratively speaking, the unit ball of a uniformly convex space is uniformly free of "flat spots". This class is more special that reflexive Banach spaces [Bre, 3.31], but includes Hilbert spaces (an easy consequence of the parallelogram identity) and L^p spaces, $p \in (1, \infty)$ [Bre, chapter 4]. Example: $\mathbb{R}^n$ is uniformly convex with l^2 norm but not with l^1 or l^∞ norms.

Theorem 36 (Radon-Riesz). *Let x_j be a sequence in a normed vector space X which converges weakly to $x \in X$. Then:*

(1) $\|x\| \leq \liminf \|x_j\|$.

(2) If X is uniformly convex then x_j strongly converges x if and only if $\|x\| = \lim \|x_j\|$.

Proof. (1) By Theorem 25 there exists $\alpha \in X^*$ such that $\|\alpha\| = 1$ and $\alpha(x) = \|x\|$. Then $\|x\| = \lim \alpha(x_j) \leq \liminf \|\alpha\| \|x_j\| = \liminf \|x_j\|$.

(2) Only if part is trivial. For the converse, putting the trivial case $x = 0$ aside, set:

$$\lambda_j := \max(\|x_j\|, \|x\|), \quad y_j := x_j / \lambda_j, \quad y := x / \|x\|.$$

Clearly, $\lambda_j \to \|x\|$ and y_j weakly converges y. By (1) we have $\|y\| \leq \liminf \|(y_j + y)/2\|$. On the other hand $\|y\| = 1$ and $\|y_j\| \leq 1$, so in fact $\|(y_j + y)/2\| \to 1$. By uniform convexity $\|y_j - y\| \to 0$, hence x_j strongly converges x. ∎

Here is an application of the Hahn-Banach Theorem to weak topologies.

Theorem 37 (Mazur). *If A is a convex subset of a LCTVS then the norm closure and weak closure of A coincide. Therefore, A is closed (respectively, dense) if and only if it is weakly closed (respectively, dense).*

Proof. Since the weak topology has less closed subsets than the original topology it follows that the weak closure $\overline{A}_w$ of A contains the original closure $\overline{A}$. To show the reverse containment assume $x \notin \overline{A}$. By Theorem 25.(4) there exist $F \in X^*$ and $K \in \mathbb{R}$ such that $\mathrm{Re}F(x) < K < \mathrm{Re}F(y)$ for every $y \in \overline{A}$. Then $\{\xi \in X : \mathrm{Re}F(\xi) < K\} = F^{-1}(\{z \in \mathbb{F} : \mathrm{Re}z < K\})$ is a weak neighborhood of x which does not intersect A, hence $x \notin \overline{A}_w$. ■

Application 38 (Mazur)**.** *Let X be a metrizable LCTVS. If x_j is a sequence in X that converges weakly to $x \in X$ then there exists a sequence y_j in X which converges strongly to x and each of its terms are convex linear combination of finitely many terms of x_j.*

Proof. Let C be the convex hull of $\{x_j\}$, namely the intersection of all convex subsets of X which contain all x_j, or equivalently the set of all convex linear combinations of finitely many terms of x_j. Then x belongs to the weak closure of C, hence to the strong closure of C by Mazur theorem. ■

5.1 Some compactness theorems

Compactness theorems are among the most useful results in analysis. Most important ones are:

- Bolzano-Weierstrass [Apo-A, 3.24]: Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.
- Tychonoff [Fol, 4.22][Mun, 37.3] (for product topology).
- Arzela-Ascoli [Fol, 4.44][Mun, 47.1] (for continuous functions or maps).
- Montel [Hör-SCV, 2.2.5] (for holomorphic functions).
- Rellich [Fol, 9.22][Tay-PDE, chapter 4] (for Sobolev functions).
- Frechet-Kolmogorov [DS, page 298][Bre, 4.26]. (for $L^p(\mathbb{R}^n)$ functions, $1 \le p < \infty$).
- Riesz [Fol, 2.30-32] (for $L^p(X, \mu)$ functions).
- Alaoglu (Theorem 39), Helley (Theorem 44), Kakutani (Theorem 45), Kakutani-Eberlein-Smulian (Theorem 47), etc. (for normed vector spaces or more generally topological vector spaces).

Here is a fundamental compactness theorem and one of the main reasons for the usefulness of weak-star topology. We will crucially use it in Chapter 9.

Theorem 39 (Alaoglu)**.** *The closed unit ball of the dual space of any normed vector space is compact in weak-star topology. More generally, if U is a neighborhood of the origin in a TVS X then $\{\alpha \in X^* : |\alpha(x)| \le 1, \forall x \in U\}$ is compact in weak-star topology.*

The closed unit ball of the dual space of a normed vector space is weak-star compact.

Proof. The closed unit ball B of the dual of normed vector space X is the set of linear elements in

$$D := \left\{ X \xrightarrow{\alpha} \mathbb{F} : |\alpha(x)| \leq \|x\|, \forall x \in X \right\} = \prod_{x \in X} D_x, \quad D_x = \{a \in \mathbb{F} : |a| \leq \|x\|\}.$$

On the other hand the weak-star topology on B and the product topology on D both coincide with the topology of pointwise convergence. (For every family X_α of topological spaces, the product topology $\prod X_\alpha$ is the weak topology generated by canonical projection maps $\pi_\alpha : \prod X_\alpha \to X_\alpha$; therefore, a net (x_i) in $\prod X_\alpha$ converges x if and only if $\pi_\alpha(x_i) \to \pi_\alpha(x)$ for every α.) Also, D is compact in product topology by Tychonoff theorem. Therefore we only need to check that B is closed in D. If (α_i) is a net in B which converges $\alpha \in D$ then for every $a, b \in \mathbb{F}$ we have

$$\alpha(ax + by) = (\lim \alpha_i)(ax + by) = \lim(a\alpha_i(x) + b\alpha_i(y)) = a\alpha(x) + b\alpha(y),$$

so α is linear, hence $\alpha \in B$. Refer [Rud-FA, 3.15] for the proof of the TVS version. ■

Application 40. *Every Banach space is (isometrically isomorphic to) a closed linear subspace of some $C(X)$, X a compact topological space.*

Proof. Consider Banach space Y and let X be the closed unit ball of Y^* equipped with the weak-star topology. X is compact by Alaoglu. Define the linear map $F : Y \to C(X)$, $F(y)(\alpha) = \langle y, \alpha \rangle$. For every $y \in Y$ we have

$$\|F(y)\| = \sup_{\|\alpha\| \leq 1} |\langle y, \alpha \rangle| = \|y\|$$

by Theorem 27. This shows that F is an isometry. The range of any isometric operator between Banach spaces is a closed subspace by a standard Cauchy sequences argument. The inverse of $F : Y \to \mathrm{Ran}_F$ is also continuous by inverse mapping theorem. ■

Next theorem gather some elementary statements about the fundamental notions of separability, reflexivity and metrizability:

Theorem 41 (Separability, reflexivity and metrizability)**.** *Let X be a normed vector space.*

(1) If X^* is separable so is X. (The converse is not true.)

(2) If X is reflexive so is every closed linear subspace of it.

(3) X is reflexive if and only if X^* is so.

(4) None of the weak or weak-star topologies on X^* are metrizable if vector space dimension of X is infinite.

(5) The weak topology on the closed unit ball of X generated by a family $\mathcal{F} \subseteq X^*$ which is separable and separates points in X, is metrizable. Specially, the weak topology on the closed unit ball of X is metrizable if X^* is separable, and that the weak-star topology on the closed unit ball of X^* is metrizable if X is separable.

(6) If X is reflexive then the weak topology on the closed unit ball of X is metrizable if X is separable.

Proof. (1) Let α_j be a dense sequence in X^*. For each j find $x_j \in X$ such that $\|x_j\| = 1$ and $\alpha_j(x_j) > \frac{1}{2}\|\alpha_j\|$. We claim that the linear span of $\{x_j\}$ is dense in X. If not, there exists $\alpha \in X^*$ such that $\|\alpha\| = 1$ and $\alpha(x_j) = 0$ for every j. For every $\epsilon > 0$ one can find α_j with $\|\alpha - \alpha_j\| < \epsilon$. This leads to the following contradiction:

$$\frac{1}{2}(1-\epsilon) < \frac{1}{2}\|\alpha_j\| < |\alpha_j(x_j)| = |(\alpha - \alpha_j)(x_j)| \leq \|\alpha - \alpha_j\| < \epsilon.$$

The converse is not true: l^1 is separable but not $l^\infty = (l^1)^*$.

(2) Let Y be a closed linear subspace of X, with $i : Y \hookrightarrow X$ the inclusion map. Let F be a continuous linear functional on Y^*. Then $X^* \to \mathbb{F}$, $\alpha \mapsto F(\alpha \circ i)$, is a continuous linear functional on X^*, so there exists $x \in X$ such that $F(\alpha \circ i) = \alpha(x)$ for every $\alpha \in X^*$. If $\alpha \in X^*$ vanishes on Y then $\alpha(x) = 0$. This shows that $x \in {}^\perp Y^\perp = \overline{Y} = Y$. Every $\beta \in Y^*$ can be extended to some $\alpha \in X^*$, namely $\beta = \alpha \circ i$, so $F(\beta) = \alpha(x) = \beta(x)$.

(3) Let X be reflexive, and fix $\mathcal{F} \in X^{***}$. Then the mapping $X \to \mathbb{F}$, $x \mapsto \mathcal{F}(\widehat{x})$, is a continuous linear functional α on X such that $\mathcal{F}(\widehat{x}) = \alpha(x)$. Since every $F \in X^{**}$ is of the form $\widehat{x}$, $x \in X$, it follows that $\mathcal{F}(F) = F(\alpha)$. This means that X^* is reflexive. Conversely, if X^* is reflexive, so is $(X^*)^*$, so is its closed linear subspace X by (2).

(4,5,6) [Roy, section 15.4]. ■

Exercise: Prove that l^1 is separable, but not l^∞. (*Hint.* Use the Cantor diagonal argument for the second.)

Example 42. $X = C([-1,1])$ is not reflexive. Here is a reason. The Dirac unit mass functionals $\delta_x : X \to \mathbb{F}$, $f \mapsto f(x)$, $x \in [-1,1]$ constitute an uncountable family of elements of X^* with $\|\delta_x - \delta_y\| = 2$ for every two distinct points $x, y \in [-1,1]$. This shows that X^* is not separable. Since X is clearly separable, if X were reflexive, then Theorem 41.(1) would imply that X^* is separable. *Another argument.* If X were reflexive then by Theorem 25.(3), assuming $\alpha \in C([-1,1])^*$ given by $\alpha(f) = \int_{-1}^0 f - \int_0^1 f$, one could find $f_0 \in C([-1,1])$ such that $\|f_0\| = 1$ and $\alpha(f_0) = \|\alpha\|$. This is absurd because $\|\alpha\| = 2$ and $|\alpha(f)| < \|f\|$ for every $f \in C([-1,1]) \setminus \{0\}$. More generally, one can prove that if Y is a compact Hausdorff space then $C(Y)$ is reflexive (respectively, separable) if and only if Y is finite (respectively, second countable) [Roy, pages 302, 251]. ■

Example 43 (von Neumann)**.** Let $A \subseteq l^2$ be the set of all $\sqrt{j}e_j$, $j = 1, 2, \ldots$, where $e_0, e_1, e_2, \ldots$ is the standard orthonormal basis of $l^2 = l^2(\{0, 1, 2, \ldots\})$. Then, the origin is in the weak closure of A, but no sequence in A weakly converges the origin. To verify the first statement, we need to check that A intersects with the basic neighborhood

$$\{y \in l^2 : |\langle x_k, y\rangle| < \epsilon, \forall k = 1, \ldots, n\}, \quad x_k = (x_{k1}, x_{k2}, \ldots) \in l^2, \quad k = 1, \ldots, n, \quad \epsilon > 0,$$

or that, equivalently, there exists positive integer j such that $\sqrt{j}\,|x_{kj}| < \epsilon$ for every k. This is true because $\sum_i |x_{ki}|^2 < \infty$ implies that $\sqrt{i}\,|x_{ki}| < \epsilon$ for every k and almost all i. For the second statement, contrapositively, assume a sequence j_n of positive integers such that $a_n := \sqrt{j_n}e_{j_n}$ weakly converges 0. Then, $\langle a_n, x\rangle \to 0$ for $x := \sum (j+1)^{-2}e_j \in l^2$ implies that $j_n \to \infty$. However, Theorem 52.(7) implies that a_n is strongly bounded. ■

Recall that a topological space is **sequentially compact** if every sequence has a convergent subsequence. In metrizable spaces, this notion coincides with the usual notion of compactness (every open cover has a finite subcover), but in general neither implies the other [Kel, page 138]. Note that Alaoglu Theorem does *not* say that the closed unit ball of the dual of a normed vector space is sequentially compact with respect to the weak-star topology. (Example: $\alpha_j : l^\infty \to \mathbb{F}$, $(x_0, x_1, \ldots) \mapsto x_j$, is a sequence in the closed unit ball of $(l^\infty)^*$ but has no weak-star convergent subsequence.); however we have:

Theorem 44 (Helley selection principle)**.** *(1) The closed unit ball of the dual space of a separable normed vector space X is sequentially compact in weak-star topology; more explicitly, every bounded sequence F_j in X^* has a subsequence F_{j_n} and $F \in X^*$ such that $F_{j_n}(x) \to F(x)$ for every $x \in X$. More generally, if U is a neighborhood of the origin in a separable TVS X then every sequence F_j in X^* which is equibounded on U (namely there exists $C > 0$ such that $|F_j(x)| \leq C$ for every j and $x \in U$) has a subsequence which converges in weak-star topology.*

(2) The closed unit ball of a reflexive normed vector space X (for example a Hilbert space or an $L^p(X, \mu)$ space, $1 < p < \infty$; no separability is assumed.) is sequentially compact in weak topology; more explicitly, every bounded sequence in X has a weakly convergence subsequence.

Equibounded sequences of continuous linear functional on separable normed vector spaces have pointwisely convergent subsequences.

Bounded sequences in reflexive normed vector spaces have weakly convergence subsequences.

Proof. (1) *TVS case.* One can assume $C = 1$. Set $K := \{F \in X^* : |F(x)| \leq 1, \forall x \in U\}$. We are supposed to show that K is sequentially compact in weak-star topology, however we know by Alaoglu theorem that K is compact in weak-star topology compact, so we are done by proving that K is metrizable in weak-star topology. Let x_k be a dense sequence of points in X. We assert that the (weak-star) topology $\mathcal{T}$ of K coincides with the topology $\mathcal{T}_d$ induced by the metric $d(F, G) = \sum 2^{-k}|F(x_k) - G(x_k)|$. Fixing F, each summand of d is $\mathcal{T}$-continuous with respect to G and the series converges uniformly (because it is dominated by $\sum 2^{-k+1}$), so d is $\mathcal{T}$-continuous, hence $\mathcal{T}_d \subseteq \mathcal{T}$. The identity map $K \to K$, where the source is equipped with topology $\mathcal{T}$ and the target with $\mathcal{T}_d$, is a bijective continuous map from a compact space to a Hausdorff one, hence a homeomorphism. We proved $\mathcal{T}_d = \mathcal{T}$.

Normed vector space case. We give a direct argument avoiding Alaoglu. Let x_k be a dense sequence of points in X. Let F_j be a sequence in X^*, which is bounded by $C > 0$. For each k the scalar sequence $\langle x_k, F_j \rangle$ is bounded in $\mathbb{F}$, so has a convergent subsequence by Bolzano-Weierstrass theorem. By Cantor diagonal argument, after passing to a subsequence, we can assume that for each k the sequence $\langle x_k, F_j \rangle$ converges. For every $\epsilon > 0$ and $x \in X$, choosing x_l with $\|x - x_l\| < \epsilon$, the estimation

$$|\langle x, F_j - F_k \rangle| \leq |\, \langle x - x_l, F_j - F_k \rangle \,| + |\, \langle x_l, F_j - F_k \rangle \,| \leq 2C\epsilon + |\, \langle x_l, F_j - F_k \rangle \,|,$$

shows that $\langle x, F_j \rangle$ converges for every $x \in X$.

(2) Let x_j be a bounded sequence in a reflexive normed vector space X. The closed linear span Y of $\{x_j\}$ is reflexive (Theorem 41.(2)) and separable (because $\mathbb{F}$ is separable). Y^* is also separable by Theorem 41.(1). $\widehat{x}_j$ is a bounded sequence of continuous linear functional on Y^*, so by (1), has a subsequence $\widehat{x}_{j_n}$ which converges pointwisely to some $\widehat{y}$, $y \in Y$. This latter statement, since every functional in X^* restricts to a functional in Y^*, means that x_{j_n} weakly converges y.

Here is a direct argument for Hilbert spaces. Let x_j be a sequence in a Hilbert space X, which is bounded by $C > 0$. For every $k \in \mathbb{N}$ the scalar sequence $\langle x_k, x_j \rangle$ is bounded in $\mathbb{F}$, so has a convergent subsequence by Bolzano-Weierstrass theorem. By Cantor diagonal argument, after passing to a subsequence, we can assume that for every k the sequence $\langle x_k, x_j \rangle$ converges, or equivalently, $\langle y, x_j \rangle$ converges for every y in the linear space Y of $\{x_j : j \in \mathbb{N}\}$. For every $\epsilon > 0$ and $y' \in \overline{Y}$, choosing $y \in Y$ with $\|y - y'\| < \epsilon$, the estimation

$$|\langle y', x_j - x_k \rangle| \leq |\langle y' - y, x_j - x_k \rangle| + |\langle y, x_j - x_k \rangle| \leq 2C\epsilon + |\langle y, x_j - x_k \rangle|,$$

shows that $\langle y, x_j \rangle$ converges for every $y \in \overline{Y}$. Therefore $y \mapsto \lim \langle y, x_j \rangle$ is a well-defined bounded linear functional on Hilbert space $\overline{Y}$. By Riesz representation theorem there exists $x \in \overline{Y}$ such that $\lim \langle y, x_j \rangle = \langle y, x \rangle$ for every $y \in \overline{Y}$. Since the same equality trivially holds for every $y \in \overline{Y}^{\perp}$ it follows that the equality holds for every $y \in \overline{Y} + \overline{Y}^{\perp} = X$. This means that x_j weakly converges x. ∎

Application 45 (Banach-Saks-Kakutani)**.** *Every bounded sequence in a uniformly convex Banach space has a subsequence whose arithmetic means is strongly convergent.*

Bounded sequences in uniformly convex Banach spaces have strongly Césaro convergent subsequences.

Proof. For general case refer [Die, chapter 8]. We prove it for Hilbert spaces. Let x_j be a sequence in Hilbert space X, bounded by $C > 0$. By Helley selection principle (Theorem 44) we can assume that x_j converges weakly to some $x \in X$. Replacing x_j by $x_j - x$ we can assume $x = 0$. After passing to a subsequence one can assume that $|\langle x_j, x_k \rangle| \leq 1/(j-1)$ for $j \geq 2$ and $1 \leq k < j$. Then for every positive integer n we have

$$\left\| \frac{1}{n} \sum_{j=1}^{n} x_j \right\|^2 = n^{-2} \left(\sum_{j=1}^{n} \|x_j\|^2 + \sum_{1 \leq k < j \leq n} 2\mathrm{Re}\, \langle x_j, x_k \rangle \right) \leq$$
$$n^{-2} \left(C^2 n + \sum_{1 \leq k < j \leq n} 2(j-1)^{-1} \right) = \frac{C^2 n + 2(n-1)}{n^2},$$

which approaches 0 as $n \to \infty$. ∎

Application 46 (Hardy functions)**.** *Suppose $p \in (1, \infty)$, $\mathbb{D} \subseteq \mathbb{C}$ the open unit disk, $d\theta$ the Lebesgue measure on unit circle $\mathbb{T} := \partial\mathbb{D}$. If f is a harmonic function on $\mathbb{D}$ such that*

$$\sup_{0<r<1} \int \left| f\left(re^{\sqrt{-1}\theta} \right) \right|^p d\theta < \infty, \tag{5.1}$$

(notation: $f \in h^p(\mathbb{D})$) then there exists $f^* \in L^p(\mathbb{T})$ such that f the Poisson extension of f^* in the sense that

$$f\left(re^{\sqrt{-1}\theta}\right) = \int f^*\left(e^{\sqrt{-1}\varphi}\right) P_r(\theta - \varphi)d\varphi,$$

where

$$P_r(t) = \sum_{j\in\mathbb{Z}} r^{|j|}e^{\sqrt{-1}jt} = \frac{1-r^2}{1-2r\cos t + r^2} = \frac{1-\left|re^{\sqrt{-1}t}\right|^2}{\left|1-re^{\sqrt{-1}t}\right|^2}.$$

Conversely, if $f : \mathbb{D} \to \mathbb{C}$ is the Poisson extension of some $f^* \in L^p(\mathbb{T})$ then $f \in h^p(\mathbb{D})$.

Proof. Let $f \in h^p(\mathbb{D})$. For any $r \in (0,1)$ let $f_r : \mathbb{T} \to \mathbb{C}$ be the restriction of f to the circle $\{|z| = r\}$, namely $f_r(\exp(\sqrt{-1}\theta) = f(r\exp(\sqrt{-1}\theta)$. The assumption (5.1) exactly says that $\{f_r : r \in (0,1)\}$ is a bounded set in $L^p(\mathbb{T})$. Fix some sequence $r_j \in (0,1)$ which approaches 1. By Helley selection principle, after passing to a subsequence, $r_j \to 1$ and f_{r_j} weakly converges some $f^* \in L^p(\mathbb{T})$ in the sense that $\lim \int f_{r_j} P d\theta = \int f^* P d\theta$ for every $P \in L^q(\mathbb{T}) = (L^p(\mathbb{T}))^*$. On the other hand each $f(r_j-)$ is a continuous function on $\overline{\mathbb{D}}$ which is harmonic on $\mathbb{D}$, so equals the Poisson extension of its boundary values [Ahl, page 169]. We have

$$f\left(re^{\sqrt{-1}\theta}\right) = \lim f\left(r_j re^{\sqrt{-1}\theta}\right) = \lim \int f_{r_j}\left(e^{\sqrt{-1}\varphi}\right) P_r(\theta-\varphi)d\varphi = \int f^*\left(e^{\sqrt{-1}\varphi}\right) P_r(\theta-\varphi)d\varphi.$$

For the converse (and to know what happens for $p = 1$ or $\mathbb{C}^m$ instead of the complex plane) refer [Rud-RCA, 11.16, 11.30][Rud-SCV, 4.3.3][Kra, chapter 8]. ■

Although weak topologies are far from being metrizable but we have the following surprising result:

Theorem 47 (Eberlein-Smulian-Kakutani)**.** *The closed unit ball B of a Banach space X is compact if and only if B is weakly sequentially compact if and only if X is reflexive.*

The closed unit ball of a Banach space X is weakly compact if and only it is weakly sequentially compact if and only if X is reflexive.

Every bounded sequence in a uniformly convex Banach space has a subsequence whose arithmetic means are strongly convergent.

Proof. [Roy, page 304][DS, chapter 5][Die, chapter 4]. The deepest part is to deduce weak compactness from sequentially weak compactness, a miracle. ■

Application 48. *Let X be a reflexive Banach space, $Y \subseteq X$ a closed convex subset and $x \in X$ a point. Then Y has a point of shortest distance to x.*

Proof. We can assume $x = 0$ and $0 \notin Y$. Set $\delta := \inf_{y \in Y} \|y\|$, and let y_j be a sequence in Y such that $\|y_j\| \to \delta$. Neglecting finitely many terms one can assume that all y_j lies in the intersection of Y with the closed ball B of radius 2δ around the origin in X. B is weakly sequentially compact by Eberlein-Smulian theorem, so after passing to a subsequence, one can assume that y_j weakly converges $y \in B$. Since $Y \cap B$ is closed and convex it is weakly closed by Mazur theorem, therefore $y \in Y$. By Theorem 25 there exists $\alpha \in X^*$ such that $\|\alpha\| = 1$ and $\alpha(y) = \|y\|$. Then:

$$\delta \leq \|y\| = \alpha(y) = \lim \alpha(y_j) \leq \lim \|\alpha\| \|y_j\| = \lim \|y_j\| = \delta.$$

So y is a point in Y of shortest distance to x. ■

Example 49. This example shows that the conclusion of Application 48 fails if reflexivity assumption is dropped. Supposed the space $X := C([-1, 1]; \mathbb{R})$, and let Y be the closed linear subspace consisting of all $y \in X$ with $\int_{-1}^{0} y = \int_{0}^{1} y = 0$. (Note that X is not reflexive by Example 42.) Fix $x \in X$ with $\int_{-1}^{0} x = 1 = -\int_{0}^{1} x$. Since $\int_{-1}^{0} x - y = 1$ it follows that $\max_{[-1,0]} x - y \geq 1$ and equality happens exactly when $x - y$ is constantly 1 on $[-1, 0]$. Similarly, $\min_{[0,1]} x - y \leq -1$ and equality happens exactly when $x - y$ is constantly -11 on $[0, 1]$. Since these two equality condition can not hold simultaneously it follows that $\|x - y\|_X > 1$. However one can really find elements y in Y such that $\|x - y\|_X$ is sufficiently closed to 1. ■

Chapter 6

Duality theory III: Krein-Milman theorem

References: [Rud-FA, 3.22-25].

Assuming a vector space X, a subset $S \subseteq X$ and a point $x \in X$, x is called an **extreme point** of S if x is not an internal point of any line interval whose end points are in S, except when both end points are in x, more precisely, for every $y, z \in S$ and every $\lambda \in (0,1)$ if $\lambda y + (1-\lambda)z \in S$ then $y = z = x$.

Theorem 50 (Krein-Milman)**.** *Let X be a TVS on which X^* separates points (for example a LCTVS or the dual of a TVS equipped with weak-star topology), and K a nonempty compact subset. Then K has at least one extreme point. If K is also convex then K is the closed convex hull of the set of its extreme points.*

Proof. [Rud-FA, 3.23]. Uses Zorn lemma together with a variation of Hahn-Banach separation theorem. ∎

Application 51 (Stone-Weierstrass theorem)**.** *Let X be a compact Hausdorff space.*

(1) Let $C(X;\mathbb{R})$ be the real algebra of continuous real-valued functions on X. A subalgebra $A \subseteq C(X;\mathbb{R})$ (subalgebra means being closed under vector space operations and the pointwise multiplication) which contains the constant function 1 and separates points (namely for every two distinct points $x, y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$) is dense.

(2) Let $C(X)$ be the complex algebra of continuous functions on compact topological space X. A subalgebra $A \subseteq C(X)$ (subalgebra means being closed under vector space operations and the pointwise multiplication) which contains the constant function 1, is closed under the pointwise conjugation, and separates points (namely for every two distinct points $x, y \in X$ there exists $f \in A$ such that $f(x) \neq f(y)$) is dense.

Proof. (1) Recall that the dual of $C(X;\mathbb{R})$ is canonically identified with the space $M(X)$ of finite signed Borel measures on X (Theorem 53). If, by contradiction, A is not dense in $C(X;\mathbb{R})$ then by Theorem 25 the set K of measures $\mu \in M(X)$ which vanishes on A (namely $\int f d\mu = 0$ for every $f \in A$) and their total variation $\|\mu\| := \int d|\mu|$ is ≤ 1 has a nonzero element. K is clearly convex and weak-star compact by Alaoglu theorem. By Krein-Milman theorem K has a extreme point ν with $\|\nu\| = 1$. The main observation is that every $g \in A$ with values in $(0, 1)$ gives the following convex representation of ν in terms of other elements in K:

$$\nu = t\frac{g\nu}{t} + (1-t)\frac{(1-g)\nu}{1-t}, \quad t := \|g\nu\| = \int g d|\nu| \in (0, 1).$$

That $g\nu/t$ and $(1-g)\nu/(1-t)$ vanish on A is because A is closed under multiplication. Since ν is an extreme point it follows that $\nu = g\nu/t$, hence g has the same value at all points of the support of ν, namely the points $x \in X$ such that $\int_V |d\nu| > 0$ for every open neighborhood V of x. From this we can easily deduce that the support of ν consists of only one point: If x, y are two distinct points in the support of ν, since A separate points one can find $g \in A$ such that $g(x) \neq g(y)$; by adding a large enough constant to g and then multiplying by a small enough positive real number one can assume that g has values in $(0, 1)$, and this is a contradiction. Therefore, ν equals the Dirac unit mass measure $\pm\delta_x$ at some $x \in X$. Since the constant function one belongs to A, we get the contradiction $0 = \int 1 d\nu = \pm 1$.

(2) Apply (1) to the algebra A' of real parts of elements of A. Note that A' contains the imaginary parts of element of A because A is closed under multiplication by $\sqrt{-1}$. ■

Krein-Milman theorem has many other applications, among them:

- Gelfand-Raikov theorem: *A locally compact group has enough irreducible unitary representations to separate points of the group.* [Fol-AHA, 3.24].
- Bochner theorem: *In a locally compact abelian group G the Fourier transform provides a one-to-one correspondence between continuous positive-definite*[1] *functions φ on G which are normalized $\varphi(1) = 1$ and probability measures μ on the Pontryagin dual group $\widehat{G}$, namely $\varphi(g) = \int_{\widehat{G}} \xi(g) d\mu(\xi)$ for every $g \in G$.* [Fol-AHA, 4.19]

[1] $\int f(x)\overline{f(y)}\varphi(y^{-1}x)dydx \geq 0$ for every $f \in L^1(G)$.

Chapter 7

Duality theory IV: Summary of results

References: [Rud-FA, chapters 3,4].

The most important slogan of functional analysis is that (continuous linear) functionals can be used to detect many notions in functions spaces. In the following theorem we gather several instances of this phenomenon.

Theorem 52. *Continuous linear functionals can be used for:*

1. *Separating convex subsets. Let A and B be two disjoint nonempty convex subsets of a TVS X. If A is open then A and B can be separated by closed hyperplanes in the sense that there exists $F \in X^*$ and $K \in \mathbb{R}$ such that $\mathrm{Re}F(a) < K \leq \mathrm{Re}F(b)$ for every $a \in A$ and $b \in B$. If A is compact, B is closed and X is locally convex then A and B can be strictly separated by closed hyperplanes in the sense that there exists $F \in X^*$ and $K_1, K_2 \in \mathbb{R}$ such that $\mathrm{Re}F(a) < K_1 < K_2 < \mathrm{Re}F(b)$ for every $a \in A$ and $b \in B$. Specially, if X is a LCTVS then X^* separate points in X in the sense that for every two distinct points x, y in X there exits $F \in X^*$ such that $F(x) \neq F(y)$.*

2. *Computing the closure of linear subspaces. If Y is a linear subspace of a LCTVS X and Z is a linear subspace of X^* then ${}^{\perp}Y^{\perp}$ gives the (norm) closure of Y in X and ${}^{\perp}Z^{\perp}$ gives the weak-star closure of Z in X^*. ($\perp$ is defined on page 42.) In words: $x \in X$ belongs to the (norm) closure of Y in X exactly when every continuous linear functional on X which vanishes on Y also vanishes on x; and $\alpha \in X^*$ belongs to the weak-star closure of Z in X^* exactly when α kills every $x \in X$ which is also killed by all members of Z.*

3. *Proving density. If Y is a linear subspace of a LCTVS X and Z is a linear subspace of X^* then Y is dense in X if and only if there is no nonzero continuous linear functional on X which vanishes on whole Y (in notations: $Y^{\perp} = \{0\}$); and Z is dense in X^* if and only if the origin is the only common zero of all elements of Z (in notations: ${}^{\perp}Z = \{0\}$).*

4. *Computing the closure of convex subsets; Mazur. If A is a convex subset of a LCTVS then the (original) closure and weak closure of A coincide.*

5. *Computing the norms of elements. If x is an element of a normed vector space X then*

$$\|x\| = \sup\{|\langle x, \alpha\rangle| : \alpha \in X^*, \|\alpha\| \leq 1\}.$$

6. *Computing the norms of operators. If $T : X \to Y$ is a bounded operator between normed vector spaces then*

$$\|T\| = \sup\{|\langle Tx, \alpha\rangle| : x \in X, \|x\| \leq 1, \alpha \in Y^*, \|\alpha\| \leq 1\}.$$

7. *Proving boundedness of subsets. A subset of a LCTVS is bounded if and only if it is weakly bounded.*

8. *Computing quotients. If X is a normed vector space and Y a closed linear subspace then we have canonical isometric isomorphisms:*

$$(X/Y)^* \xrightarrow{\cong} Y^\perp, \quad \alpha \mapsto (x \mapsto \alpha(x+Y)),$$

$$X^*/Y^\perp \xrightarrow{\cong} Y^*, \quad \alpha + Y^\perp \mapsto (y \mapsto \alpha(y)).$$

9. *Proving continuity of linear maps. If $T : X \to Y$ is a linear map between normed vector spaces then T is continuous if and only if $T^*(Y^*) \subseteq X^*$ (namely $\beta \circ T$ is continuous for every continuous linear functional β on Y) if and only if T is weak-to-weak continuous (namely continuous if both X and Y are equipped with weak topologies).*

10. *Computing kernels of operators. If $T : X \to Y$ is a bounded operator between normed vector then*

$$\mathrm{Ker}_T = {}^\perp\mathrm{Ran}_{T^*}, \quad \mathrm{Ker}_{T^*} = \mathrm{Ran}_T^\perp,$$
$$\overline{\mathrm{Ran}_{T^*}}^{w*} = \mathrm{Ker}_T^\perp, \quad \overline{\mathrm{Ran}_T} = {}^\perp\mathrm{Ker}_{T^*}.$$

11. *Proving that operators are injective or range-dense. If $T : X \to Y$ is a bounded operator between normed vector spaces then:*

    (a) *T is range-dense if and only if T^* is injective.*

    (b) *T is injective if and only if T^* is weak-star range dense.*

12. *Proving surjectivity of operators. If $T : X \to Y$ is a bounded operator between Banach spaces, then the followings are equivalent:*

    (a) *T is surjective.*

    (b) *T is open. Equivalently, there exists $r > 0$ such that $TX_1 \supseteq Y_r$, where X_r denotes the open ball of radius r in X around the origin, and similarly for Y_r. More concretely, for every $y \in Y$ there exists $x \in X$ with $Tx = y$ and $\|x\| \leq r^{-1}\|y\|$.*

    (c) *There exists $r > 0$ such that $\overline{TX_1} \supseteq Y_r$.*

(d) *T^* is bounded from below; equivalently, T^* is injective and closed-range. (Theorem 8.)*

13. *Proving that operators are closed-range. If $T : X \to Y$ is a bounded operator between Banach spaces then the followings are equivalent:*

    (a) *T is range-closed.*

    (b) *There exists $C > 0$ such that for every $y \in \mathrm{Ran}_T$ there exists $x \in X$ with $Tx = y$ and $\|x\| \leq C\|y\|$.*

    (c) *T^* is weak-star range-closed.*

    (d) *T^* is range-closed.*

    (e) *$\|Tx\| \geq C \inf\{\|x - \xi\| : \xi \in \mathrm{Ket}_T\}$ for some $C > 0$ and every $x \in X$.*

Proof. (1) Proved in Theorem 25.(4).

(2) $\overline{Y} \subseteq {}^{\perp}Y^{\perp}$ and $\overline{Z}_{w*} \subseteq {}^{\perp}Z^{\perp}$ are clear. If $x \in X \setminus \overline{Y}$ then by Theorem 25.(4) there exists a continuous linear functional F on X which vanishes on $\overline{Y}$ (hence on Y) but not at x, namely $x \notin {}^{\perp}Y^{\perp}$. If $F \in X^* \setminus \overline{Z}_{w*}$ then by Theorem 25.(4) there exists a weak-star-continuous linear functional α on X^* which vanishes on $\overline{Z}_{w*}$ (hence on Z) but not at F. By Theorem 34, α is of the form $\widehat{x}$, $x \in X$, acting by $F \mapsto F(x)$. We have proved that $\widehat{x} \notin {}^{\perp}Z^{\perp}$.

(3) Immediate from (2).

(4) Proved in Theorem 37.

(5, 6) Proved in Theorem 27.

(7) Only if part is trivial. The converse is a deep theorem whose proof needs Alaoglu, Hahn-Banach separation theorem and Baire category theorem [Rud-FA, 3.18]. (It also follows from a theorem of Mackey on dual systems [Tre, 36.2][MV, pages 248-9].) Here we prove the special case when X is a normed vector space. Let $S \subseteq X$ be a weakly bounded subset of normed vector space X. This means that for each $\alpha \in X^*$ there exists $C_\alpha > 0$ such that $|\langle s, \alpha\rangle| \leq C_\alpha$ for every $s \in S$. In terms of the natural embedding $X \to X^{**}$, $x \mapsto \widehat{x}$, $\widehat{x}(\alpha) = \langle x, \alpha\rangle$, this exactly means that the family of bounded linear functionals $\widehat{s} : X^* \to \mathbb{F}$, $s \in S$, is pointwisely equibounded; hence it is uniformly equibounded by the uniform boundedness principle (Theorem 5.(1)): There exists $C > 0$ such that $\|\widehat{s}\| < C$ for every $s \in S$. Since $\|\widehat{s}\| = \|s\|$ (Theorem 27) this means that S is bounded.

(8) The first map is clearly well-defined and linear. Its set-theoretic inverse is given by $\beta \mapsto (x + Y \mapsto \beta(x))$. Just using definitions it is straightforward to check that both these maps are isometry. In regard to the second identification, the map given is clearly well-defined and linear. Its set-theoretic inverse is: $f \in Y^*$ is mapped to $F + Y^{\perp}$, where F is any extension of f to X. (There is at least one by Theorem 25.(2$'$).) Clearly, $\|F + Y^{\perp}\| = \inf_{g \in Y^{\perp}} \|F + g\| \geq \|f\|$. The reverse inequality is immediate from 25.(2$'$) by choosing F with $\|F\| = \|f\|$.

(9) If T is continuous and $\beta \in Y^*$ then clearly $T^*\beta = \beta \circ T$ is continuous. If $T^*(Y^*) \subseteq X^*$, $\beta \in Y^*$ and x_α is a net in X weakly converging x then $\langle x_\alpha, T^*\beta\rangle \to \langle x, T^*\beta\rangle$, or equivalently, $\langle Tx_\alpha, \beta\rangle \to \langle Tx, \beta \circ T\rangle$, namely Tx_α weakly converges Tx, and this means that T is weak-to-weak continuous. Finally, assume that T is weak-to-weak continuous but not continuous. Then the image of the unit ball under T is not bounded, so is not

weakly bounded by (7). This means that there exists $\beta \in Y^*$ such that $\{| \langle Tx, \beta \rangle | : \|x\| \leq 1\}$ is not bounded. To get a contradiction it suffices to show that $T^*\beta \in X^*$. Let x_α be a net in X which weakly converges x. Then by our assumption Tx_α weakly converges Tx. Therefore $\beta \circ Tx_\alpha$ converges $\beta \circ Tx$. This means that $T^*\beta$ is weakly continuous on X. $T^*\beta \in X^*$ by Theorem 34.

(10)

$$x \in \mathrm{Ker}_T \leftrightarrow Tx = 0 \underset{\text{Theorem 25}}{\leftrightarrow} \langle Tx, \beta \rangle = 0, \forall \beta \in Y^* \leftrightarrow \langle x, T^*\beta \rangle = 0, \forall \beta \in Y^* \leftrightarrow x \in^{\perp} \mathrm{Ran}_{T^*}.$$

$$\beta \in \mathrm{Ker}_{T^*} \leftrightarrow T^*\beta = 0 \leftrightarrow \langle x, T^*\beta \rangle = 0, \forall x \in X \leftrightarrow \langle Tx, \beta \rangle = 0, \forall x \in X \leftrightarrow \beta \in \mathrm{Ran}_T^{\perp}.$$

This proves the first two equations. The last two follow from the first two and (2).

(11) Immediate from (10).

(12) (12a)⇔(12b)⇔(12c) was proved during the proof of the open mapping theorem (Theorem 5.(2)).

(12b)⇔(12d) Suppose that $TX_1 \supseteq Y_r$ for some $r > 0$. Then for every $\beta \in Y^*$ we have

$$\|T^*\beta\| = \sup_{x \in X_1} \left| \langle x, T^*\beta \rangle \right| = \sup_{x \in X_1} \left| \langle Tx, \beta \rangle \right| \geq \sup_{y \in Y_r} \left| \langle y, \beta \rangle \right| = r\|\beta\|.$$

Therefore T^* is bounded from below. Conversely, suppose that $\|T^*\beta\| \geq r\|\beta\|$ for some $r > 0$ and every $\beta \in Y^*$. We assert that $\overline{TX_1} \supseteq rY_1$. Fix $y_0 \notin \overline{T(X_1)}$. By Theorem 33 there exists $\beta \in Y^*$ such that $| \langle y_0, \beta \rangle | > 1$ and $| \langle y, \beta \rangle | \leq 1$ for all $y \in \overline{T(X_1)}$. Then

$$\|T^*\beta\| = \sup_{x \in X_1} | \langle x, T^*\beta \rangle | = \sup_{x \in X_1} | \langle Tx, \beta \rangle | \leq 1,$$

hence

$$r < r| \langle y_0, \beta \rangle | \leq r\|y_0\|\|\beta\| \leq \|y_0\|\|T^*\beta\| \leq \|y_0\|,$$

namely $y_0 \notin rY_1$.

(13) All parts are proved by applying (12) to appropriate restriction maps.

(13a)⇔(13b) If Ran_T is closed then the restriction $S : X \to \mathrm{Ran}_T$ of T to its range is a surjective continuous map between Banach space, hence an open map by (12). This means that there exists $C > 0$ such that $S(X_r) \supseteq Y_{Cr} \cap \mathrm{Ran}_T$ for every $r > 0$, where X_r is the open ball in X of radius r around the origin, and similarly for Y_r. This is exactly (13b). The converse is via a famous Cauchy sequence argument: If $Tx_j \to y$ then Tx_j is Cauchy, and by (13b) one can assume that x_j is bounded and Cauchy, so $x_j \to x$; therefore $y = Tx$.

(13a,13b)⇒(13c) Since $\overline{\mathrm{Ran}_{T^*}}^{w*} = \mathrm{Ker}_T^{\perp}$ by (10), we need to show that $\mathrm{Ker}_T^{\perp} \subseteq \mathrm{Ran}_{T^*}$. In other words, fixing $\alpha \in \mathrm{Ker}_T^{\perp}$ we need to find $F \in Y^*$ such that $\alpha = F \circ T$. Since $\alpha \in \mathrm{Ker}_T^{\perp}$ it follows that the linear functional $f : \mathrm{Ran}_T \to \mathbb{F}$, $Tx \mapsto \alpha(x)$, is well-defined. f is continuous by (13b), hence can be extended continuously to some $F \in Y^*$. Clearly, $\alpha = F \circ T$.

(13c)⇒(13d) Trivial.

(13d)⇒(13a) Let $S : X \to Z$ be the restriction of T to $Z := \overline{\mathrm{Ran}_T}$. Note that S^* is injective by (11). Every $f \in Z^*$ can be extended to some $F \in Y^*$ by Hahn-Banach theorem, so

$$\langle x, T^*F \rangle = \langle Tx, F \rangle = \langle Tx, f \rangle = \langle x, S^*f \rangle, \quad \forall x \in X,$$

hence $\mathrm{Ran}_{S^*} = \mathrm{Ran}_{T^*}$ is closed. Therefore $S^* : Z^* \to \mathrm{Ran}_{S^*}$ is open by (12). This means that there exists $r > 0$ such that $\|S^* f\| \geq r\|f\|$ for every $f \in Z^*$. (Note that S^* is injective.) Therefore S is surjective by (12), hence $\mathrm{Ran}_T = \mathrm{Ran}_S = Z = \overline{\mathrm{Ran}_T}$ is closed.

(13a)⇔(13e) Consider the naturally defined operator $S : X/\mathrm{Ker}_T \to \overline{\mathrm{Ran}_T}$, $x + \mathrm{Ker}_T \mapsto T(x)$. Note that $\mathrm{Ran}_S = \mathrm{Ran}_T$. Clearly: T is range-closed if and only if S is range-closed if and only if S is surjective if and only if S is bounded from below. This last sentence is exactly the condition in (13e). ■

We have developed some duality. To put them into action in concrete areas of analysis we need the computations of dual spaces:

Theorem 53. *(1) The dual of a Hilbert space X is isometrically isomorphic to X via $X \to X^*$, $x \mapsto \langle x, -\rangle$.*

(2) The dual of $L^p(X,\mu)$, $1 < p < \infty$, (X,μ) measure space, is isometrically isomorphic to $L^q(X,\mu)$, $1/p + 1/q = 1$, via $L^q \to (L^p)^*$, $f \mapsto \int f - d\mu$. The same is true for $p = 1$ if the measure is σ-finite.

(3) The dual of Bergman space $L^p_a(D) := L^p(D) \cap \{\text{holomorphic}\}$, $1 \leq p < \infty$, D an open subset of $\mathbb{C}^m$ equipped with Lebesgue measure, is isometrically isomorphic to $L^q_a(D)$, $1/p + 1/q = 1$, via $L^q_a(D) \to L^p_a(D)^*$, $f \mapsto \int f - d\mu$.

(4) The dual of $C(X)$, X compact Hausdorff space, is isometrically isomorphic to $M(X)$ (the vector space of complex Radon measures, normed by total variation $\|\mu\| = |\mu|(X)$) via $M(X) \to C(X)^*$, $\mu \mapsto \int -d\mu$.[1]

(4′) The dual of $C([0,1])$ is isometrically isomorphic to $NBV([0,1])$ (the vector space of functions $[0,1] \to \mathbb{C}$ which are of bounded variation $\|f\|_{NBV} := \sup \sum_{j=1}^n |f(x_j) - f(x_{j-1})| < \infty$, supremum taken over all $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, which vanish at 0 and which are continuous from the left on $(0,1)$) via Riemann-Stieltjes integration $BV([0,1]) \to C(X)^*$, $\varphi \mapsto \int -d\varphi$.

Proof. (1) Theorem 11.

(2) [Fol, 6.15].

(3) [Zhu-FT, 2.12].

(4) [Fol, 7.17][Rud-RCA, 6.19][DS, pages 262-5 and 258].

(4′) [Fol, 3.29][Dou, 1.37]. ■

Dual of $L^\infty(X,\mu)$ and many other function spaces is given in [DS, pages 375-9]. Dual of $M(X)$ is hard to describe [DS, footnote of page 374][Kpl].

[1] The real version is: The dual of $C(X;\mathbb{R})$ is isometrically isomorphic to the vector space of all signed Radon measures. Refer [Fol, Chapter 7] for the definition of Radon measures.

Chapter 8

Compact and Fredholm operators

References: [Dou, chapter 5][Rud-FA, chapter 4][Con-FA, chapter 11].

One of the most useful notions in many areas of geometry and algebra is *deformation*. This chapter develops a deformation (or homotopy) theory for operators. The role of infinitesimals is played by the so-called compact operators, and Fredholm operators are those which are invertible up to addition by an infinitesimal. (These ideas eventually lead to the K-theory for operator algebras [Bla].) Here is another motivation for compact operators. If one thinks of functional analysis as a generalization of linear algebra to infinite dimensions, then the immediate class of operators to be studied after finite rank operators (those whose range is of finite dimension) is the class of those operators which are norm limits of finite ranks; these are compact operators (at least on Hilbert spaces).

Theorem 54. *For a bounded operator T on a Banach space X, the followings are equivalent:*

(1) The image of any bounded subset of X under T has compact closure. Equivalently, if x_j is a bounded sequence in X then Tx_j has a convergent subsequence.

(2) T is weak-to-strong continuous in the sense that if x_j is a weakly convergent sequence in X then Tx_j is convergent.

In case any of the conditions above holds, T is called a **compact** *operator.*

Theorem 55. *For a bounded operator T on a Hilbert space X the followings are equivalent:*

(1) T is compact.

(2) T is the norm limit of a sequence of finite rank operators.

(3; **Schmidt** *(or* **canonical***)* **representation***) T can be represented as a norm convergent series of the form $\sum_{n=1}^{N} \lambda_n \langle \varphi_n, - \rangle \psi_n$, where $N \in \mathbb{N} \cup \{\infty\}$, $\lambda_n \in \mathbb{R}$, $\lambda_n \to 0$, and $\{\varphi_n\}, \{\psi_n\}$ are orthonormal sets.*

(T) Ran_T contains no closed infinite dimensional linear subspace.

Proposition 56 (Compacts as infinitesimals)**.** *If T is a compact operator on a separable Hilbert space X, then for every $\epsilon > 0$, there exists a finite dimensional linear subspace $Y \subseteq X$ such that the norm of the restriction of T to $Y^{\perp}$ (namely $\|PTP\|$ where P is the orthogonal projection onto $Y^{\perp}$) is less than ϵ.*

Example 57. Here are the most famous examples of compact operators:

1. A diagonal operator $l^2(\mathbb{N}) \to l^2(\mathbb{N})$, $(x_j) \mapsto (\lambda_j x_j)$ is compact if and only if $\lambda_j \to 0$.

2. All linear maps between finite dimensional normed vector spaces are compact. According to Theorem 2.(5), the identity map on a normed vector space X is compact if and only if X is of finite vector space dimension.

3. Pseudodifferential operators of negative order on closed smooth manifolds are compact. (Closed means compact and without boundary.)

4. Hilbert-Schmidt and trace class operators are compact [Arv-S, pages 70-74]. ■

Theorem 58 (Riesz's theory for compact operators)**.** *Let T be a compact operator on an infinite dimensional Banach space X. Then:*

(1) $\sigma(T)$ accumulates only at origin, so specially it is countable and contains 0.

(2) Suppose $\lambda \in \sigma(T)\backslash\{0\}$. Then λ is an eigenvalue of T of finite multiplicity (namely $\dim \mathrm{Ker}_{T-\lambda} < \infty$); λ (and also $\overline{\lambda}$) is an eigenvalue of T^* with the same multiplicity. More generally, not only the ordinary eigenspace $\mathrm{Ker}_{T-\lambda}$ but also the generalized eigenspace $\{x \in X : (T-\lambda)^n x = 0, \exists n \in \mathbb{N}\}$ is finite dimensional, and of the same dim as $\{x \in X : (T^* - \overline{\lambda})^n x = 0, \exists n \in \mathbb{N}\}$.

If, furthermore, T is normal then:

(3) $\mathcal{H}$ has an orthonormal basis of eigenvectors of T, and eigenspaces corresponding to distinct eigenvalues are orthogonal to each other.

If, furthermore, T is self-adjoint then:

(4) Eigenvalues are real and in the closed interval from $-\|T\|$ to $\|T\|$, with at least one of these endpoints being an eigenvalue.

Proof. [Rud-FA, pages 103-111]. ■

Theorem 59. *For a bounded operator T on a Banach space X, the followings are equivalent:*

(1) T is invertible modulo compacts, namely, there exists a bounded operator S on X such that $TS - 1$ and $ST - 1$ are both compact. S is called a **parametrix** *for T.*

(2) The kernel Ker_T and cokernel $\mathrm{Coker}_T = X/\mathrm{Ran}_T$ of T are both finite-dimensional. If so, then T is range-closed.

In case any of the conditions above holds, T is called a **Fredholm** *operator of* **index** $\mathrm{ind}(T) := \dim \mathrm{Ker}_T - \dim \mathrm{Coker}_T$.

Example 60. Here are the most famous examples of Fredholm operators:

1. An invertible operator plus a compact is Fredholm of index 0.

2. The unilateral forward shift operator is Fredholm of index -1.

3. Elliptic pseudodifferential operators on closed smooth manifolds are Fredholm. Their index is given by the Atiyah-Singer index theorem [Tay-PDE, chapters 7,10][AS, AS-I]. ■

Recall from linear algebra that a square matrix A is either invertible ($\det A \neq 0$) or it is neither injective nor surjective ($\det A = 0$). In other words, either $Ax = b$ has a unique solution for every b, or $Ax = 0$ has a nonzero solution; in this latter case, $Ax = b$ has a solution if and only if $b \in \mathrm{Ran}_A = \mathrm{Ker}_{A^*}^{\perp}$, and if $Ax_0 = b$ then the whole solution space of $Ax = b$ is $x_0 + \mathrm{Ker}_A$. Here is a generalization:

Theorem 61 (Fredholm alternative)**.** *Let T be a bounded operator on a Hilbert space $\mathcal{H}$ which is the sum of an invertible operator (for example, the identity map) and a compact. Then:*

(1) T is a Fredholm operator of index zero. Conversely, every Fredholm operator of index zero is an invertible plus a compact.

(2) T is range-closed (hence $\mathrm{Ran}_A = \mathrm{Ker}_{A^*}^{\perp}$) and it is injective if and only if it is surjective.

(3) Either $Ax = b$ has a unique solution for every b or $Ax = 0$ has a nonzero finite dimensional solution space $N := \ker A$; in this latter case $Ax = b$ has a solution if and only if b is orthogonal to the finite dimensional space Ker_{A^*}, and if $Ax_0 = b$ then the whole solution space of $Ax = b$ is $x_0 + N$.

Proof. (1)

(2) Immediate from (1).

(3) Immediate from (1,2). ■

Theorem 62. *Two Fredholm operators on a Hilbert space are homotopic if and only if they have the same index.*

Here is a surprising application of the Fredholm alternative:

Application 63 (Dirichlet problem; Hilbert)**.** *[RSz] or [Fol-PDE].*

Chapter 9

Spectral theory I: Commutative Banach and C* algebras

References: [Rud-FA, chapters 10-11][Dou, chapters 2,4][Fol-AHA, chapter 1].

A (unital) **Banach algebra** (or **B-algebra**) is a Banach space A which is also a $\mathbb{C}$-algebra with (multiplicative) identity element 1 such that $\|1\| = 1$ and $\|xy\| \leq \|x\|\|y\|$ for every $x, y \in A$. A (unital) **C*-algebra** is a Banach algebra A, equipped with a conjugate-linear map $A \to A$, $x \mapsto x^*$, called **involution**, such that

$$\|x^*\| = \|x\|, \quad x^{**} = x, \quad (xy)^* = y^*x^*, \quad \|xx^*\| = \|x\|^2,$$

for every $x, y \in A$.

Example 64. Here are the most important examples of B and C* algebras.

1. Let G be a locally compact Hausdorff topological group, and dx denotes its left Haar measure [Fol, 11.8]. Then the Banach space $L^1(G)$ is a Banach algebra if multiplication is defined via convolution $(f*g)(x) = \int_G f(xy^{-1})g(y)dy$. (Recall the Young inequality: Suppose $1 \leq p, q, r \leq \infty$ such that $1/p+1/q = 1+1/r$. Then the convolution of $f \in L^p$ with $g \in L^q$ is a almost everywhere defined function h in L^r satisfying $\|h\|_r \leq \|f\|_p\|g\|_q$. The proof in [Fol, 8.9] for $G = \mathbb{R}^n$ works for the general case.) For more about these algebras refer [Kan].

2. $C(X)$, X a compact Hausdorff space, is a C*-algebra with involution given by pointwise conjugation $f^*(x) = \overline{f(x)}$. Every commutative C*-algebra is of this form (Theorem 72).

3. $B(X)$, X a Hilbert space, is a C*-algebra with involution given by the adjoint of operators (Theorem 12).

4. $A(K)$, $K \subseteq \mathbb{C}$ compact, the set of continuous functions $K \to \mathbb{C}$ which are holomorphic in the interior of K is a closed subalgebra of $C(K)$, so a Banach algebra. If K is the closed unit disk then $A(K)$ is called the **disc algebra**.

5. H^∞

6. Multiplier algebras. [Kan, 1.4] ■

Theorem 65 (Fundamental lemma)**.** *Let A be a Banach algebra and $x \in A$.*

(1) If $\|x\| < 1$ then $1 - x$ is invertible, with inverse given by $\sum_{j\geq 0} x^j$.

(2) If $\|x\| < |\lambda|$, $\lambda \in \mathbb{C}$, then $\lambda - x$ is invertible, with inverse given by $\sum_{j\geq 0} \lambda^{-j-1} x^j$.

(3) The set of invertible elements of A is open, and the map $x \mapsto x^{-1}$ is continuous on it.

Let A be a Banach algebra. A **multiplicative linear functional** on A is a linear functional $\varphi : A \to \mathbb{C}$ such that $\varphi(1) = 1$ and $\varphi(xy) = \varphi(x)\varphi(y)$ for every $x, y \in A$. From this one can deduce that $\|\varphi\| = 1$. *Proof.* $\|\varphi\| \geq 1$ because $\varphi(1) = 1$. By contradiction assume $x \in A$ such that $|\varphi(x)| > \|x\|$. Since x can be written (uniquely) as $\lambda 1 - y$ where $\lambda \in \mathbb{C}$ and $y \in \mathrm{Ker}_\varphi$ it follows that $1 > |1 - z|$ where $z = y/\lambda \in \mathrm{Ker}_\varphi$. By the fundamental lemma there exists $w \in X$ which $zw = 1$. This leads to the contradiction $1 = \varphi(1) = \varphi(zw) = \varphi(z)\varphi(w) = 0$. Q.E.D. The set M_A of all multiplicative linear functionals on A, quipped with the weak star topology, is called the **maximal ideal space** of A. Note that M_A is a closed subset of the closed unit ball of the dual A^* of A, so compact by Alaoglu theorem. Let x be an element of Banach algebra A. The **spectrum** of x, denoted by $\sigma(x)$ or more precisely by $\sigma_A(x)$, is the set of all $\lambda \in \mathbb{C}$ such that $\lambda - x$ is invertible. The complement of the spectrum of x is called the **resolvent set** of x. The **spectral radius** of x, denoted by $\rho(x)$ or more precisely by $\rho_A(x)$, is the supremum of all $|\lambda|$ such that $\lambda \in \sigma(x)$.

Theorem 66 (Gelfand)**.** *Let A be a Banach algebra with maximal ideal space M_A. Then:*

(1) The spectrum of $x \in A$ is nonempty and compact.

(1; Gelfand-Mazur) If A is a division algebra (namely every nonzero element in A is invertible) then there is a unique isometric isomorphism $A \to \mathbb{C}$.

(1; Gelfand-Beurling formula) The spectral radius of x is not larger than $\|x\|$. In fact, it is given by $\lim \|x^j\|^{1/j}$.

(1) The Gelfand transform $X \to C(M_X)$ is a $\mathbb{C}$-algebra homomorphism which preserves $*$ and is contractive in the sense that $\|\widehat{x}\| \leq \|x\|$ for every $x \in A$.

(1; Spectral mapping theorem) The spectrum of each $f(x)$, $x \in A$, f an entire function, equals the image of $\sigma(x)$ under f.

If A is commutative then:

(1) $\varphi \mapsto \mathrm{Ker}_\varphi$ provides a one-to-one correspondence between element of M_A and the maximal ideals of A. (A nonempty subset $I \subseteq A$ is an ideal if it is closed under subtraction and multiplication by elements of A.) The inverse is given by $A \to A/\mathfrak{m}$, $x \mapsto x + \mathfrak{m}$. Note that maximal ideals are closed.

(1) M_A is nonempty.

(1) $x \in A$ is invertible if and only if its Gelfand transform $\widehat{x} \in C(M_A)$ is invertible.

(1) The spectrum and spectral radius of each $x \in A$ can be computed by its Gelfand transform $\widehat{x} \in C(M_A)$ in the following way:

$$\sigma_A(x) = \sigma_{M_A}(\widehat{x}) = (\text{range of } \widehat{x}), \quad \rho_A(x) = \rho_{M_A}(\widehat{x}) = (\text{uniform norm of } \widehat{x}).$$

Example 67 (Spectrum)**.** Here are the most important examples of spectrums of elements of Banach algebras:

1. The spectrum of a square matrix, as a linear map between finite dimensional vector spaces, is the set of its eigenvalues.

2. The spectrum of $f \in C(X)$ equals its range.

3. The spectrum of the unilateral forward shift (Example 13) is the closed unit disc.

4. The spectrum of $f \in L^\infty(X, \mu)$ equals its **essential range** namely the set of all $\lambda \in \mathbb{C}$ such that the inverse image of any disc of radius $\epsilon > 0$ around λ has strictly positive measure.

5. The spectrum of the multiplication operator $M_f \in B(L^2(X, \mu))$ associated to the bounded symbol $f \in L^\infty(X, \mu)$ equals the essential range of f.

6. The spectrum of the Toeplitz operator $T_f \in B(H^2(\mathbb{S}^1))$ associated to the continuous symbol $f \in C(\mathbb{S}^1)$ is given by the range of f union with the set of all $z \in \mathbb{C}$ such that the winding number of f around z is nonzero.

7. The max ideal space of $A(K)$ (Example 73) is K. [Kan]

8. $L^\infty(X)$ has the fame of having very huge and complicated max ideal space. The reason is that its elements are not true functions, but just equivalence classes of functions.

9. Let X be a completely regular Hausdorff space. The maximal ideal space of $BC(X)$ (Example 1) can be identified with the Stone-Čech compactification $\beta(X)$ of X [Mun, section 38]. ■

Example 68 (Empty maxiaml ideal spaces and spectra)**.** (1) There are Banach algebras with empty maximal ideal space.

(2) There are unbounded operators with empty spectrum. ■

Example 69. The maximal ideal space of $X := l^1(\mathbb{Z})$ (multiplication is given by convolution $(f * g)(j) = \sum_{k\in\mathbb{Z}} f(k-j)g(k)$) can be identified with the unit circle via $\mathbb{S}^1 \to M_X$, $z \mapsto \varphi_z$, given by $\varphi_z(f) = \sum_{j\in\mathbb{Z}} f(j)z^j$ for every $f \in X$. ■

Application 70 (Wiener)**.** *Let f be a nowhere-zero continuous funtion on the unit circle. If the Fourier series of f is absolutely convergent then so is the Fourier series of $1/f$.*

Theorem 71 (Šilov)**.** *Let A be a Banach algebra, $B \subseteq A$ a closed subalgebra, and $x \in B$. Then $\sigma_B(x)$ is obtained from $\sigma_A(x)$ by filling some (possibly no) holes in $\sigma_A(x)$, namely by taking union of $\sigma_A(x)$ with some of the bounded components of its complement.*

Theorem 72 (Gelfand-Naimark)**.** *(1) Every commutative unital C*-algebra is C*-isomorphic to some $C(X)$, X a compact topological space. More specifically, for every commutative unital C*-algebra A the Gelfand map $A \to C(M_A)$, $a \mapsto \widehat{a}$, $\widehat{a}(\varphi) = \varphi(a)$, is a C*-isomorphism.*

(2) For every compact Hausdorff space X the Gelfand map $X \to M_{C(X)}$, $x \mapsto \widehat{x}$, $\widehat{x}(f) = f(x)$, is a homeomorphism.

(3) The category of commutative unital C-algebras and C*-homomorphisms between them is equivalent to the category of compact Hausdorff spaces and continuous maps between them, via Gelfand maps:*

$$X \to M_{C(X)}, \quad x \mapsto \widehat{x}, \quad \widehat{x}(f) = f(x),$$

$$A \to C(M_A), \quad a \mapsto \widehat{a}, \quad \widehat{a}(\varphi) = \varphi(a).$$

(4) The category of commutative C-algebras and C*-homomorphisms between them is equivalent to the category of locally compact Hausdorff spaces and proper continuous maps between them via Gelfand maps.*

A **von Neumann algebra** (or **W*-algebra**) is a C*-subalgebra of some $B(X)$, X a Hilbert space, which is closed under weak operator topology.

Example 73. Here are the most important examples of W* algebras.

1. $L^\infty(X, \mu)$ is a W*-algebra with involution given by pointwise conjugation. Every commutative W*-algebra is of this form [Dav, II.2.9].

2. Double commutant. ■

Chapter 10

Spectral theory II: Bounded normal operators

References: [Rud-FA, chapters 4,12][Con-FA, chapters 2,9][Fol-AHA, chapter 1].

Theorem 74 (Spectral theorem; bounded normal operators)**.** *Let T be a bounded normal operator on a separable Hilbert space $\mathcal{H}$. Then:*

1. *Multiplication operator version. T is unitarily equivalent to a multiplication operator; more precisely, there exists a σ-finite measure space (X, μ), an essentially bounded Borel measurable function $f \in L^\infty(X, \mu)$ and a unitary operator $U : \mathcal{H} \to L^2(X, \mu)$ such that $T = U^{-1} M_f U$, where M_f acts on $L^2(X, \mu)$ by $g \mapsto fg$. If $\mathcal{H}$ is separable then μ can be assumed finite.*[1]

2. *Direct integral decomposition version There exists a unique σ-finite measure μ on $\sigma(T)$ and a unitary operator U from $\mathcal{H}$ to the direct integral $\int_{\sigma(T)}^{\oplus} \mathcal{H}_\lambda d\mu(\lambda)$ such that $T = U^{-1} M_\lambda U$, where M_λ acts on the direct integral by mapping the section $s(\lambda)$ to the section $\lambda s(\lambda)$. Uniqueness is in the sense that if another representation is given by $\int_{\sigma(T)}^{\oplus} \mathcal{H}'_\lambda d\mu'(\lambda)$, after modifying μ and μ' to make* **multiplicity functions** $\lambda \mapsto \dim \mathcal{H}_\lambda$ *and $\lambda \mapsto \dim \mathcal{H}'_\lambda$ nowhere zero, it is the case that μ and μ' are mutually absolutely continuous and the multiplicity functions are almost everywhere the same.*

3. *Projection-valued measure version. There exists a unique projection-valued measure P defined on the Borel sigma algebra of the spectrum $\sigma(T)$ of T with values in projections on $\mathcal{H}$ such that $T = \int_{\sigma(T)} \lambda dP(\lambda)$.*

4. *Continuous functional calculus. The maximal ideal space of the (commutative) C^*-algebra C_T^* generated by T is homeomorphic to $\sigma(T)$, hence the Gelfand transform $C_T^* \to C(\sigma(T))$ is a $*$-isomorphic isomorphism. In other words, there exists a unique map $\Phi : C(\sigma(T)) \to B(\mathcal{H})$ with the following properties:*

[1] Neither X is the spectrum of T, nor f is the coordinate function (on $\mathbb{C}$); however both of these are true if T is cyclic (or multiplicity-free).

(a) Φ *preserves the* $\mathbb{C}$*-algebra structure, conjugation and norm.*

(b) *The identity map is mapped to* T*.*

(c) *The spectrum of each* $\Phi(f)$ *equals the image of* $\sigma(T)$ *under* f*. This is called spectral mapping theorem.*

(d) *Each* $\Phi(f)$ *is a nonnegative operator if* f *is a nonnegative function.*

(e) *If* f_j *is a sequence of functions converging in* $C(\sigma(T))$ *to* f *then* $\Phi(f_j)$ *converges* $\Phi(f)$ *in norm.*

5. *Bounded Borel functional calculus. There exists a positive regular Borel measure* μ *supported on* $\sigma(T)$ *and a* $*$*-isomorphic isomorphism* Ψ *from the von Neumann algebra* W_T^* *generated by* T *to* $L^\infty(\sigma(T), \mu)$ *which extends the Gelfand map* $C_T^* \to C(\sigma(T))$ *of (4). In other words, there exists a unique map* $\Psi : L^\infty(\sigma(T), \mu) \to B(\mathcal{H})$ *with the following properties:*

   (a) Ψ *preserves the* $\mathbb{C}$*-algebra structure, conjugation.*

   (b) Ψ *is contractive namely* $\|\Psi(f)\| \leq \|f\|$ *for every* $L^\infty(\sigma(T), \mu)$*.*

   (c) *The identity map is mapped to* T*.*

   (d) *Each* $\Phi(f)$ *is a nonnegative operator if* f *is a nonnegative function.*

   (e) *If* f_j *is a sequence of functions converging almost everywhere to* f *and* $\|f_j\|$ *is bounded then* $\Psi(f_j)$ *converges* $\Psi(f)$ *in the strong operator topology.*

Theorem 75 (Spectral multiplicity theorem; bounded normal operators)**.** *Let* T_j*,* $j = 1, 2$*, be bounded normal operators on separable Hilbert spaces* $\mathcal{H}_j$*.*

(1) Assume a direct integral representations for T_j *as Theorem 74.(2) with measure* μ_j *chosen such that the Hilbert space dimension of* $(\mathcal{H}_j)_\lambda$ *is nonzero for* μ_j*-almost every* λ*. Then* T_1 *and* T_2 *are unitarily equivalent if and only if they have the same spectrum, measures* μ_1 *and* μ_2 *are mutually absolutely continuous, and the multiplicity functions* $\lambda \mapsto \dim(\mathcal{H}_j)_\lambda$ *are almost everywhere the same.*

(2) Assume a projection-values measure representations for T_j *as Theorem 74.(3) with projection-valued measure* P_j*. Then* T_1 *and* T_2 *are unitarily equivalent if and only if they have the same spectrum and* $P_1 = P_2$*.*

Chapter 11

Spectral theory III: Unbounded normal operators

References: [Rud-FA, chapter 13][Wei, chapters 4-5][dOl, chapters 1-2].

The applications of functional analysis to partial differential equations is through the language of unbounded operators, which we develop in this chapter.

11.1 Unbounded operators

Let X and Y be Hilbert spaces over $\mathbb{C}$. By an **unbounded operator** $A : X \to Y$ we just mean a $\mathbb{C}$-linear map $A : \mathrm{Dom}_A \to Y$ defined on some linear subspace $\mathrm{Dom}_A \subseteq X$. (So every bounded (= continuous) operator is an unbounded operator in this terminology!) A is called **densely defined** if Dom_A is dense in X.

1. A is called **closed** if the graph $\mathcal{G}_A = \{(f, Af) : f \in \mathrm{Dom}_A\}$ of A is closed in $X \times Y$. Equivalently, for every sequence f_j in Dom_A such that f_j converges to $f \in X$ and Af_j converges to g we must have $f \in \mathrm{Dom}_A$ and $Af = g$. (The closed graph theorem says that an unbounded operator defined on whole X is closed if and only it is continuous, but when $\mathrm{Dom}_A \neq X$ neither of the notions of closedness and continuity implies the other.)

2. If A is densely defined then the **adjoint** of A, denoted by A^*, is the unbounded operator $A^* : Y \to X$ defined as follows: Dom_{A^*} consists of all $g \in Y$ such that $\langle Ah, g\rangle$ is continuous with respect to $h \in \mathrm{Dom}_A$, namely $|\langle Ah, g\rangle| \leq C\|h\|$ for some positive constant C. If so then the functional $\mathrm{Dom}_A \to \mathbb{C}$ mapping h to $\langle Ah, g\rangle$ has a unique continuous extension to X by Hahn-Banach theorem, so by Riesz representation theorem there exists a unique $f \in X$ such that $\langle Ah, g\rangle = \langle h, f\rangle$, and we set $A^*g = f$. Equivalently, A^* can be characterized by $\mathcal{G}_{A^*} = (J\mathcal{G}_A)^\perp$ where $J(f, g) = (g, -f)$.

3. If A is densely defined then A^* is closed.

   (*Proof.* $\mathcal{G}_{A^*} = (J\mathcal{G}_A)^\perp$ and the orthogonal complement of every subset of a Hilbert space is closed.)

4. If A is densely defined and closed then so is A^*, and we have $A^{**} = A$.

   (*Proof.* Since $J^2 = -\mathrm{id}$ and J commutes with the operations of closure and orthogonal complement when applied to subspaces it follows that $J\mathcal{G}_{A^*}^{\perp} = -\overline{\mathcal{G}_A} = \mathcal{G}_A$. To show that A^* is densely defined assume $g \in \mathrm{Dom}_{A^*}^{\perp}$. Since $(0, g) \in J\mathcal{G}_{A^*}^{\perp} = \mathcal{G}_A$ it follows that $g = 0$. This shows that Dom_{A^*} is dense in Y. Finally, $\mathcal{G}_{A^{**}} = J\mathcal{G}_{A^*}^{\perp} = JJ\mathcal{G}_A^{\perp\perp} = -\overline{\mathcal{G}_A} = \mathcal{G}_A$ shows that $A^{**} = A$.)

5. If A is densely defined then $\mathrm{Ran}_A^{\perp} = \mathrm{Ker}_{A^*}$. If A is densely defined and closed then $\mathrm{Ran}_{A^*}^{\perp} = \mathrm{Ker}_A$, so Ker_A is closed.

   (*Proof.* The first assertion is immediate from definition. Replacing A by A^* gives the second.)

6. Staying in the framework of Zermelo-Frankel set theory (not using the axiom of choice) one can not construct a noncontinuous unbounded operator $X \to Y$ which is defined on whole X [Wri][Fol, page 179]. In other words, all concrete noncontinuous unbounded operators are partially defined. The most important examples of unbounded operators are differential operators, specially the d-bar operator in our case.

11.2 Spectral theorem

Theorem 76 (Spectral theorem; unbounded normal operators)**.**

11.3 Closed range theorem for unbounded operators

Recall that in finite dimensional linear analysis (namely linear algebra) we have $\mathrm{Ran}_A = \mathrm{Ker}_{A^*}^{\perp}$ for every matrix A, so that $Au = f$ has a solution if and only if $\langle f, g\rangle = 0$ for every g with $A^*g = 0$. In infinite dimenional linear analysis (namely functional analysis) we have only $\overline{\mathrm{Ran}_A} = \mathrm{Ker}_{A^*}^{\perp}$ for densely defined closed operators A. The following theorem says how to deal with the closure in the left hand, and gives an if and only if condition for the solvability of $Au = f$.

Theorem 77 (Closed range theorem for unbounded operators)**.** *Let $A : X \to Y$ be a densely defined closed unbounded operator between Hilbert spaces. Then:*

(1) For every $f \in Y$, there exists $u \in X$ with $Au = f$ if and only if $|\langle f, g\rangle| \le C\|A^*g\|$ for every $g \in \mathrm{Dom}_{A^*}$ and some $C \ge 0$.

(2) For every closed subspace $F \subseteq Y$ which $F \supseteq \mathrm{Ran}_A$, we have $F = \mathrm{Ran}_A$ if and only if $\|g\| \le C\|A^*g\|$ for every $g \in \mathrm{Dom}_{A^*} \cap F$ and some $C \ge 0$.

(3) Ran_A is closed if and only if $\|g\| \le C\|A^*g\|$ for every $g \in \mathrm{Dom}_{A^*} \cap \overline{\mathrm{Ran}_A}$ and some $C \ge 0$.

(4) Ran_A is closed if and only if Ran_{A^*} is closed.

Proof. (1) We only prove the if part because the other direction trivial. In accordance with the general philosophy of the duality theory in functional analysis (namely understanding a linear space through linear functionals living on it) one observes that our desired u is exactly the element of X which represents the anti-linear functional $\mathrm{Ran}_{A^*} \to \mathbb{C}$

mapping A^*g to $\langle f, g\rangle$. This functional is well-defined and bounded by C according to our hypothesis. By Hahn-Banach theorem it can be extended to a linear functional on whole X with the same bounded operator norm. (Another way: First extend by continuity to $\overline{\mathrm{Ran}_{A^*}}$ and then extend to whole X by declaring the functional to vanish on the orthogonal complement of $\overline{\mathrm{Ran}_{A^*}}$.) If $u \in X$ is the vector that represents this extended functional according to the Riesz representation theorem then $\langle f, g\rangle = \langle u, A^*g\rangle$ for every $g \in \mathrm{Dom}_{A^*}$. It then follows by the very definition of the adjoint that $u \in \mathrm{Dom}_{A^{**}}$ and $A^{**}u = f$. Since A is densely defined and closed it follows that $A^{**} = A$, and we are done.

(2) For the if part, fixing arbitrary $f \in F$ and $g \in \mathrm{Dom}_{A^*}$, according to (1) we need to show that $|\langle f, g\rangle| \leq C\|A^*g\|$ for some $C \geq 0$. Let $g = g' + g''$, $g' \in F$, $g'' \in F^\perp$, be the orthogonal decomposition of g. Since $F \supseteq \mathrm{Ran}_A$ it follows that $F^\perp \subseteq \mathrm{Ran}_A^\perp = \mathrm{Ker}_{A^*}$, hence we deduce $g'' \in \mathrm{Ker}_{A^*}$, $g' \in \mathrm{Dom}_{A^*}$ and $A^*g' = A^*g$. By applying our hypothesis to g' we have

$$|\langle f, g\rangle| = |\langle f, g'\rangle| \leq \|f\|\|g'\| \leq C\|f\|\|A^*g\|.$$

Only if part. If for some $g \in \mathrm{Dom}_{A^*} \cap F$ we have $A^*g = 0$, then $g = Af \in \mathrm{Ran}_A = F$ for some $f \in \mathrm{Dom}_A$ and $A^*Af = 0$, hence $\|g\|^2 = \langle Af, Af\rangle = \langle f, A^*Af\rangle = 0$, therefore $g = 0$. As a result we need only show that

$$G := \{g/\|A^*g\| : g \in \mathrm{Dom}_{A^*} \cap F, A^*g \neq 0\}$$

is bounded as a subset of the Hilbert space F. For every $h = Af \in \mathrm{Ran}_A = F$ the set $\{\langle h, g\rangle : g \in G\}$ is a bounded subset of $\mathbb{C}$ with bound $\|h\|$. This means that G is weakly bounded in F. It is famous that weakly bounded subsets of Hilbert spaces are bounded. (This is immediate from the uniform boundedness principle [Fol, 5.13]. Refer [Jos-RS, page 85] for a direct proof. Compare [Rud-FA, 3.18].)

(3) In (2) let F be the closure of the range of A.

(4) It suffices to prove the only if part because the other direction can be deduced from this one by replacing A with A^*. Let Ran_A be closed. Then (3) gives

$$\|g\| \leq C\|A^*g\|, \quad \forall g \in G, \quad G := \mathrm{Dom}_{A^*} \cap \overline{\mathrm{Ran}_A}. \tag{11.1}$$

This inequality combined with a straightforward Cauchy sequence argument shows that A^* restricted to G has closed range. (Details: Assume a sequence $g_j \in G$ such that A^*g_j converges to $f \in X$. Since A^*g_j is Cauchy it follows from (11.1) that g_j is also Cauchy, hence convergent to some $g \in Y$. Since A^* has closed graph it follows that $g \in G$ and $A^*g = f$.) However the range of $A^*|_G$ equals the range of A^* because A^* kills the orthogonal complement of $\overline{\mathrm{Ran}_A}$. This proves the only if part. ■

Chapter 12

Implicit function theorem for Banach spaces

References: [Jos-A, chapter 10]

One of the most useful observations in advanced calculus is that:

1. If real variable y depends smoothly on real variable x, $y(x_0) = y_0$ and $dy/dx(x_0) \neq 0$, then x depends smoothly on y locally around y_0.

2. If $F(x, y) = 0$ is an implicit smooth equation between real variables x and y, $F(x_0, y_0) = 0$ and $\partial F/\partial y(x_0, y_0) \neq 0$ then y is given by a smooth function $y = y(x)$ around x_0. For example, the equation of circle $x^2 + y^2 = 1$ can be solved locally around each of its points with $y > 0$ (respectively, $y < 0$) by $y = \sqrt{1 - x^2}$ (respectively, $y = -\sqrt{1 - x^2}$).

More generally we have:

Theorem 78 (Inverse and implicit function theorems)**.** *(1) Assuming a C^k map $F : U \to \mathbb{R}^n$, $k \in \{0, 1, \ldots, \infty\}$, defined on an open $U \subseteq \mathbb{R}^n$, if the $n \times n$ Jacobian matrix of F is invertible at a point $x_0 \in U$ then F is a C^k diffeomorphism (namely, it is bijective and with C^k inverse) on some neighborhood of x_0.*

(2) Assuming the natural splitting $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$ coordinated by (x, y), a C^k map $F : U \to \mathbb{R}^n$, $k \in \{0, 1, \ldots, \infty\}$, defined on some open of $\mathbb{R}^{m+n}$, and a point $(x_0, y_0) \in U$ such that $F(x_0, y_0) = 0$, if the $n \times n$ Jacobian matrix $\partial F/\partial y$ is invertible at (x_0, y_0) then there exist neighborhoods $V \subseteq \mathbb{R}^m$ around x_0 and $W \subseteq \mathbb{R}^n$ around y_0 such that for every $x \in V$ there exists a unique $y = f(x) \in W$ such that $F(x, y) = 0$. Furthermore, $f : V \to W$ is C^k.

These two statements are equivalent: Applying (1) to $(x, y) \mapsto (x, F(x, y))$ gives (2). Applying (2) to $x \mapsto y - F(x)$ gives (1). Here is the sketch of the proof of (1). Without loss of generality one can assume $x_0 = 0$, $F(x_0) = 0$ and the Jacobian matrix of F at x_0 is the identity matrix. The intuition here is that F behaves like identity maps around the origin. Local injectivity is immediate from the mean value theorem for differentiation, but to prove local surjectivity we need to use the Banach fixed point theorem: *A map $T : X \to X$ on a Banach space X which is contractive (namely $\|Tx - Ty\| \le a\|x - y\|$ for some $0 < a < 1$ and every $x, y \in X$) has a unique fixed point.*

In this short chapter we formulate and prove an infinite-dimensional analogue of Theorem 78. First we need to make sense of differentiability in infinite-dimensional spaces. Let X and Y be normed vector spaces. A map $F : U \to Y$ defined on an open $U \subseteq X$ is said to be **differentiable** at $x \in U$ if there exists a continuous linear map $T : X \to Y$ such that $\|F(x+\xi) - F(x) - T(\xi)\|/\|\xi\| \to 0$ as $\xi \to 0$. If so, then T is unique, denoted by $dF(x)$, and called the **total** (or **Frechet**) **derivative** of F at x. F is called differentiable on U if it is differentiable at every point of U. If F is differentiable on U then it is called **twice differentiable** at $x \in U$ if $U \to B(X; Y)$, $x \mapsto dF(x)$, is differentiable at x. If so, the total derivative of this latter map is denoted by $d^2F(x)$; it is initially an element of $B(X; B(X; Y))$, but can be naturally identified as a bilinear map $X \times X \to Y$ bounded in the sense that $\|d^2F(x)(\xi, \xi')\| \le C\|\xi\|\|\xi'\|$ for some $C > 0$ and every $\xi, \xi' \in X$. Higher order derivatives are defined inductively. If F is differentiable on U then it is called C^1 if $U \to B(X; Y)$, $x \mapsto dF(x)$, is continuous. If F is twice differentiable on U then it is called C^2 if $U \to B(X; B(X; Y))$, $x \mapsto d^2F(x)$, is continuous. C^k, $k \in \{0, 1, \ldots, \infty\}$, is defined inductively.

Theorem 79 (Inverse and implicit function theorems for Banach spaces)**.** *(1) Let X and Y be Banach spaces. Assuming a C^1 map $F : U \to Y$ defined on an open $U \subseteq X$, if the Frechet derivative $dF : X \to Y$ is invertible (as a map between Banach spaces) at a point $x_0 \in U$ then F is a C^1 diffeomorphism on some neighborhood of x_0, namely there exist opens $x_0 \in V \subseteq U$ and $W = F(V) \subseteq Y$ such that the restriction $F : V \to W$ is bijective and with C^1 inverse.*

(2) Let X, Y and Z be Banach spaces, and let (x, y) coordinate $X \times Y$. Assuming a C^1 map $F : U \to Z$ defined on open $U \subseteq X \times Y$ and a point $(x_0, y_0) \in U$ such that $F(x_0, y_0) = 0$, if $\partial F/\partial y : Y \to Z$ is invertible at (x_0, y_0) then there are neighborhoods $V \subseteq X$ around x_0 and $W \subseteq Y$ around y_0 such that for every $x \in V$ there exists a unique $y = f(x) \in W$ such that $F(x, y) = 0$. Furthermore, $f : V \to W$ is C^1.

Proof. (1)

(2) Apply (1) to $(x, y) \mapsto (x, F(x, y))$. ■

For more about implicit function theorems refer [KP]. Richard Hamilton (the inventor of the Ricci flow) developed an implicit function theorem for special classes of Frechet spaces appearing in differential geometry [Ham].

Chapter 13

GNS construction

References: [Con-FA, chapter 8].

Theorem 80 (Gelfand-Naimark-Segal)**.** *Every C^*-algebra is $*$-isomorphic to a closed C^*-subalgebra of some $B(X)$, X a Hilbert space.*

Proof. ■

With the same ideas one can prove a generalization of GNS, the so-called Stinespring's dilation theorem which characterizes completely positive maps from C*-algebras into $B(X)$ in terms of $*$-homomorphisms into $B(Y)$, Y some other Hilbert space [Pau, chapter 4].

Chapter 14

Semigroups

References: [Rud-FA, chapter 13][Lax, chapter 34][Bre, chapter 7].

Recall that the first-order linear constant-coefficient system of ordinary differential equations $dx/dt = Ax$, A a square matrix of complex numbers, can be solved using matrix exponentials by $x(t) = \exp(At)x(0)$. In this chapter we generalize this to infinitely many equations.

A **(one-parameter) semigroup** T of operators on a Banach space X is a family $T_t : X \to X$, $0 \le t < \infty$, of bounded operators on X such that $T_0 = 1$ and $T_{t+s} = T_t \circ T_s$ for every $t, s \ge 0$. The **infinitesimal generator** of T is the unbounded operator $A : \mathrm{Dom}_A \to X$ defined by $A(x) = \lim_{t\to 0+}(T_t x - x)/t$. T is called **strongly continuous** if $\|T_t x - x\| \to 0$ as $t \to 0+$ for every $x \in X$. Here is a basic fact:

Theorem 81. *Let T be a strongly continuous semigroup in a Banach space X with the infinitesimal generator A. Then:*
(1) A is densely defined and closed.
(1) $\|T_t\| \le M\exp(\omega t)$ for some $\omega \in \mathbb{R}$ and $M > 0$.
(1) $dT_t x/dt = AT_t = T_t A$ for every $x \in \mathrm{Dom}_A$.
(1) $\mathrm{Dom}_A = X$ if and only if $T_t \to 0$ in the norm topology as $t \to 0+$ if and only if A is bounded and $T_t = \exp(tA)$.

Theorem 82 (Hille-Yoshida)**.** *(1) An unbounded operator A on a Banach space X generates a strongly continuous semigroup T with $\|T_t\| \le M\exp(\omega t)$ for some $\omega \in \mathbb{R}$ and $M > 0$ if and only if A is densely defined and closed, every $\lambda > \omega$ belongs to the resolvent set of A, and $\|(\lambda - A)^{-n}\| \le M(\lambda - \omega)^{-n}$ for every positive integer n.*

(2) An unbounded operator A on a Banach space X generates a strongly continuous semigroup T with $\|T_t\| \le M$ for some $M > 0$ if and only if A is densely defined and closed, every $\lambda > 0$ belongs to the resolvent set of A, and $\|(\lambda - A)^{-1}\| \le M(\lambda - \omega)^{-1}$.

Theorem 83 (Stone)**.** *If U is a strongly continuous semigroup of unitary operators on a Hilbert space X then there exists a self-adjoint unbounded operator H on X such $U_t = \exp(\sqrt{-1}Ht)$.*

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